AN ECONOMIC ORDER QUANTITY MODEL FOR

KNOWN DEMAND-WITH-TREND

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Richard Edwin Hazlett

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GLOSSARY OF ESSENTIAL SYMBOLS

General Symbols

t	decision period
T	forecast period (decision periods/forecast period)
U	unit price (dollars/unit)
S	ordering cost (dollars/unit)
i	decision period carrying cost (percent/unit/period)
I	forecast period carrying cost (percent/unit/period)
(S/iU)	decision period cost parameter ratio
(S/IU)	forecast period cost parameter ratio
k	square root of twice the decision period cost parameter ratio
K	square root of twice the forecast period cost parameter ratio

Symbols Used Under Known-Constant Demand

d	decision period demand (units/period)
D	forecast period demand (units/period)
Q^*	economic order quantity, Harris Model (units/order)
n^*	economic number of decision period orders (orders/period)
N^*	economic number of forecast period orders (orders/period)
(tvc)	decision period total variable costs (dollars/period)
(TVC)	forecast period total variable costs (dollars/period)

Symbols Used Under Known Demand-With-Trend

B	basic demand (units/period)

A	trend in demand (units/period) or (percent of the basic demand/period)
d_t	decision period demand (units/period)
D_t	forecast period demand (units/period)
m_t^*	economic trend-correction factor (units/period)
$Q_t'^*$	economic order quantity, under proper application of Harris Model (units/period)
$(TVC)'^*$	forecast period total variable costs of policy based upon $Q_t'^*$ (units/period)
M^*	economic, integer-valued, constant-approximation to the trend-correction factor (units/period)
$Q_M'^*$	economic order quantity based upon M^* (units/period)
$(TVC)'_{M^*}$	forecast period total variable costs of policy based upon Q'_{M^*} (dollars/period)
Q_F'	false basic economic order quantity (units/period)
$(TVC)'_F$	forecast period total variable cost of policy based upon Q_F' (dollars/period)
M_P^*	predicted value of M^* (units/period)
$Q'_{M_P^*}$	economic order quantity based upon M_P^*, Trend EOQ (units/period)
$(TVC)'_{M_P^*}$	forecast period total variable costs of policy based upon $Q'_{M_P^*}$ (dollars/period)
(DEV)	deviation between predicted and actual values of M^* (units/period)

SUMMARY

The primary objective of this study was to develop a dynamic inventory model, or decision rule, for determining basic economic order quantities under known demand-with-trend. The investigation was accomplished under the remaining basic strictures of the Harris EOQ Model. These strictures are as follows:

1. Cost minimization is the criterion for the determination of the economic order quantity.
2. Replenishment of stock is instantaneous.
3. Shortages, or stockouts, are not allowed.
4. Unit costs, carrying costs, and ordering costs are known and constant.

The secondary objective of this study was to investigate the possibility of incorporating certain selected modifications of the Harris Model within the framework of the Trend EOQ Model.

A linear step-function was selected to represent known demand-with-trend. This function assumes that, if the forecast period demand is known-with-trend, the decision period will be chosen so that its demand will be constant within any given decision period. This assumption was made to enable application of the Harris Model to determine those economic order quantities, for each decision period, which would yield the lowest-possible forecast period total variable costs. This cost was used in the evaluation of the effectiveness of the model.

The two main premises which guided the direction of the investigation were: (1) an economic order quantity equation could be found

which have the same form as the demand equation and (2) the ordering policy based upon this equation, or step-function, would closely approximate the lowest-possible forecast period total variable cost ordering policy.

The concept of a linear order quantity step-function was investigated under the additional stricture that all order quantities must be integer-valued. It was shown that there is one value among all possible values of the trend-correction factor such that its ordering policy yields the lowest forecast period total variable cost among all such policies. A direct mathematical solution for determining this economic trend-correction factor was found to be impractical. Instead, eighty-one combinations of values of the basic demand, the trend in demand, the cost parameter ratio, and the forecast period were selected, and an integer-valued, constant-approximation to the economic trend-correction factor was determined for each observation. These observations were used to develop a linear-logarithmic multiple regression equation for predicting the integer-valued, constant-approximation to the economic trend-correction factor.

It was found that the regression equation would be sufficiently accurate to predict the economic trend-correction factor within a few integer-units. The forecast period total variable costs of ordering policies developed using the Trend EOQ Model should not differ significantly from the lowest-possible forecast total variable costs, which were developed by applying the Harris Model within each decision period. The Trend EOQ Model forecast period total variable costs should, however, be significantly lower than those resulting from the application of the

Harris Model within the entire forecast period (i.e., ignoring the trend in demand).

The Trend EOQ Model directly accommodates the Non-Inventory EOQ Model, under conditions of known demand-with-trend. The Trend EOQ Model does not directly accommodate the Production EOQ Model. Quantity Discount EOQ Models could be accommodated within the framework of the Trend EOQ Model, but the number of additional calculations required may increase the study and the implementation costs to the extent that they exceed the expected cost savings to be derived from the application.

CHAPTER I

INTRODUCTION

The Basic Economic Order Quantity Model

The first mathematical concept of inventory theory was presented in 1915, by Ford W. Harris (1). Harris' model gave an economic order quantity through the balancing of two costs: the cost of carrying stock and the cost of ordering stock. These two costs are called "opposing" costs, since one will increase and the other decrease as the order quantity varies. In its original form, Harris' model cannot receive wide application. The restrictive conditions of its assumptions are rarely found in practical inventory problems. The five basic assumptions behind Harris' model are:

(1) demand is constant and known with certainty;
(2) cost minimization is the criterion for determing the economic order quantity;
(3) replenishment of stock is instantaneous (i.e., lead time is zero or known with certainty);
(4) no shortages are allowed (i.e., the cost of a shortage is infinite); and
(5) unit costs, carrying costs, and ordering costs are known and constant.

The value of Harris' model must not be underestimated, however, since prior to it there were no published procedures for inventory control with a rational, mathematical basis. Since 1915, the model has served as a basic building block for more realistic inventory models and has become invaluable in the teaching of inventory theory.

Using Harris' model as a beginning, researchers have modified or relaxed its assumptions to develop models which are suitable for a wider range of inventory problems. Many modifications have been made to fit the model to specific practical situations.

The simplicity of the balancing of opposing costs, through the use of graphical methods or calculus, makes the Harris model a desirable concept for an introduction to the theory of inventory models. Insufficient knowledge of the limitations of the model, however, will lead to its indiscriminate use. Once a sound background has been obtained, academic progress to more complex models should be rapid.

Modifications of the Basic Economic Order Quantity Model

The majority of the modifications of Harris' model have involved the removal of one or more of the last three basic assumptions. The scope of the model has thus been broadened, but still within the stricture of a known and constant demand.

Demand may be divided into four major patterns, as is shown below:

(1) Class I - constant demand;
(2) Class II - demand with trend;
(3) Class III - seasonal demand; and
(4) Class IV - seasonal demand with trend.

The Harris model will accomodate Class I demand. However, the three remaining classes of demand comprise a larger percentage of the demands encountered in practice than does Class I. Further broadening of the scope of the economic order quantity models would be accomplished with their adaptation to cover other demand patterns.

Objectives of the Study

The primary objective of this study shall be to develop a dynamic inventory model or decision rule for determining basic economic order quantities for Class II demands. The investigation will be accomplished under the strictures of Harris' model, except that demand will no longer be constant and known with certainty.

The secondary objective shall be to investigate the possibility of incorporating certain selected modifications of the last three basic assumptions of the Harris model within the framework of the model or decision rule developed for Class II demands. These selected modifications will be:

(1) receipt of stock is not instantaneous, but occurs over some finite time period;
(2) the balancing of ordering and shortage costs;
(3) one or more unit prices are given for ranges of the order quantity.

The Use of Economic Order Quantity Models

The Role of the Economic Order Quantity

The two major inventory control systems are: the Q-system and the P-system. The Q-system requires that the size of an order be fixed, but allows the frequency of orders to vary. The P-system requires the order period to be fixed, but allows the order quantity to vary.

The economic order quantity is the fixed parameter of the Q-system. One method of determining the fixed parameter of the P-system is to divide the economic order quantity by the demand. Thus the economic order quantity is a basis for the two major inventory control systems.

When Economic Order Quantity Models Can be Applied

The feasibility of applying an economic order quantity model to a specific inventory problem depends upon the availability of: a model covering the exact situation and the values of the parameters required by the model.

The determination of the specific model to use is made by consideration of the nature of the parameters and by selection of a suitable decision criterion (e.g., cost minimization). A model may already exist which uses the selected criterion and encompasses all the required parameters; some modification of an existing model may have to be made; or an entirely new model may have to be developed to fit the given problem.

Once a model has been chosen, the values of the cost and demand parameters must be determined. Sales records and forecasts, inventory records, and accounting records would be used to obtain these values. If the records did not give the exact values, estimates would have to be made from the data or by management.

When Economic Order Quantity Models Should be Applied

The practicability of an economic order quantity model application depends upon the ratio of the cost savings to the cost of the study and implementation.[1] The accuracy and the magnitude of this ratio should be considered whenever a decision is to be made about the implementation of an inventory program.

The estimate of the cost of an inventory study and its implementation can be made with reasonable accuracy. The accuracy of the cost savings to be obtained from the application depends upon the accuracy of the parameter values and the sensitivity of the model to changes in these values (2, pp. 176-181).

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[1]The cost savings should be expressed in terms of their present worth.

The magnitude of the ratio of cost savings to study and implementation costs can be expressed either in terms of a percentage or the time in which the inventory policy would take to pay for itself. The implementation decision would be made after a comparison of the magnitude of the ratio and the investment objectives of management. The ratio is simply the efficiency of the inventory study, with the study and implementation costs as the input and the cost savings as the output. If the proposed program does not violate existing "taboos" and management can be expected to act in a rational manner, the more efficient the study the greater the chance that management will institute the program. (Rather than discuss the rationality of management, it will be assumed that the study should be made as efficient as possible.)

One method of increasing the efficiency of the study is by use of a Distribution-By-Value analysis or an A-B-C classification scheme (3, pp. 19-22). The greatest cost savings, in an application of economic order quantity models, result from the determination of economic order quantities for each individual item of those items with the highest dollar usage (i.e., Class A items). Assuming that the cost of calculating an economic order quantity is constant for every item, the marginal rate of the efficiency of the study will decrease with the decrease in dollar usage. This decrease may be offset slightly by a grouping of similar items of low dollar usage and the determination of an economic order quantity for each group, rather than for each item.

Applications Under Known Demand-With-Trend

If one of the assumptions of a model is violated, either knowingly or unknowingly, the costs of the policy thus developed will not be the lowest possible minimum costs. One violation would occur when the known demand exhibits a trend. This deviation from the lowest possible minimum costs, or the sensitivity of the economic order quantity model to changes in the demand parameter, will be investigated for two cases. The deviation in these cases are:

(1) cost deviations between the policy which falsely assumes demand to be constant and the "optimum" policy, and,
(2) cost deviations between the policy determined by the model developed, assuming a Class II demand, and the "optimum" policy.

The "optimum" policy will be considered the policy with the lowest - possible minimum cost.

CHAPTER II

THE DEVELOPMENT OF THE ECONOMIC ORDER QUANTITY MODEL

In this chapter, the brief history of inventory control literature will be discussed. The mathematical derivation of Harris' model will be presented, in a notation consistent with that to be used in developing an economic order quantity model for known demand-with-trend. Modifications of the Harris model and its basic assumptions will be presented, along with their intended applications.

The History of Inventory Control Literature

Prior to the publication of Harris' model in 1915, the literature on inventory control was generally of a qualitative nature. The advent of a quantitative procedure for determining order size must surely have caused the businessmen and academicians of the period to give some thought to developing mathematical tools for inventory control. The first major outbreak of literature on the subject, however, did not begin until the middle 1920's. Articles by Cooper, Davis, Pennington, and Wilson all discussed the determination of economic order quantities in a manner similar to that of Harris. (4), (5), (6), (7), (8).

Academic interest in inventory control grew during the 1930's and 1940's, until there were many groups concerned with the development of the theory: industrial engineers, economists, operations researchers, industrial managers, and mathematicians. In spite of some feelings of encroachment upon sacred grounds among some of these groups, the last two decades have brought a wealth of valuable literature.

Probably the two most valuable areas of theoretical progress have been in the substitution of probabilistic demand concepts for the earlier deterministic concepts and the associated development of methods for determining safety stocks. In 1953, Whitin summarized and expanded the existing theory along these lines. (9) Around this time, the elaborate articles of Arrow, Harris, and Marschak and Dvoretzky, Kiefer, and Wolfowitz appeared to further refine the theory. (10), (11), (12)

Dynamic inventory concepts for fluctuating patterns of demand have been developed through the use of servomechanism concepts and dynamic programming. (13), (14)

Articles on applications of inventory control theory to specific practical problems became more numerous during the 1950's. Bibliographies by Whitin and Hanssmann give insight into the diversity of these applications. (9), (15)

The first years of this decade saw the publication of several texts on inventory theory designed for both practical and academic use. This was mainly a time for reconsideration, evaluation, and summarization of past developments. Excellent texts were published by Fetter and Dalleck, Miller and Starr, and Hadley and Whitin. (16), (2), (18) These texts should provide adequate discussions of the phases of inventory control not explicitly treated within this paper.

Presentation of Harris' Model

The Behavior of Inventory Under Harris' Assumptions

An inventory-behavior-pattern may be developed using Harris' basic assumptions. This pattern is shown in Figure 1.

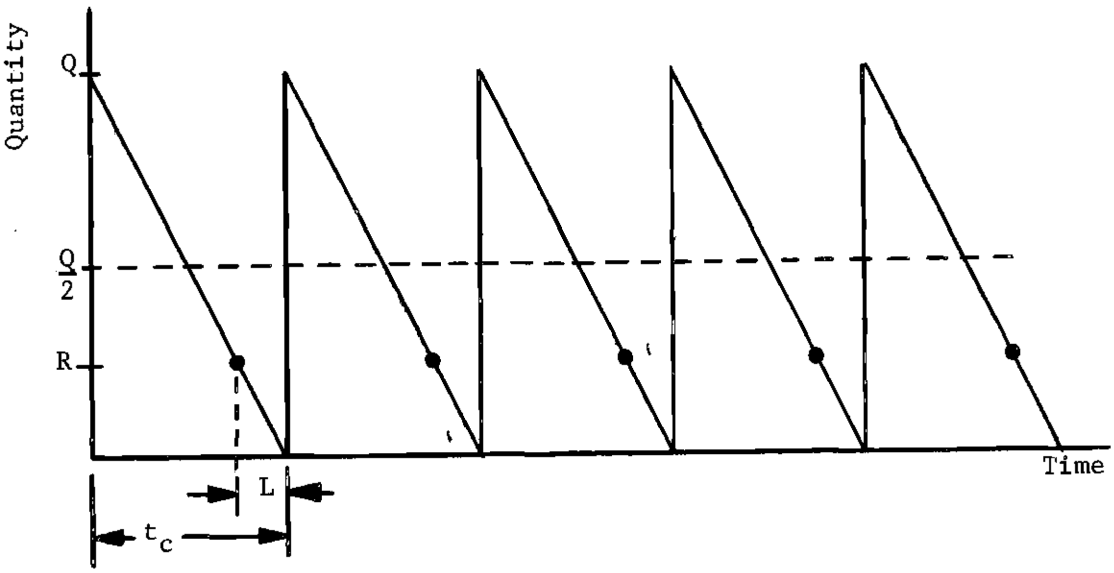

Figure 1. Inventory-Behavior-Pattern: Harris' Model.

At the beginning of the time period, an order quantity, Q, is placed in inventory; depletion of inventory occurs uniformly over time; when the reorder point, R, is reached, an order is placed for Q units; upon exhaustion of the inventory, Q units are received and immediately placed in inventory; and the cycle repeats itself.

Under this behavior-pattern, it can be seen that the average inventory level is one-half the order quantity, or Q/2. At no time is the inventory level allowed to become negative. (i.e., no shortages or stockouts are allowed.)

The reorder point level, R, is equal to the product of the lead time, L, and the depletion, or demand, rate. The inventory cycle time, t_c, may be determined by dividing the order quantity by the demand. The number of orders, N, necessary to fill the entire demand, may be determined by taking the reciprocal of t_c or by dividing the demand by the order quantity.

Determination of the Economic Order Quantity

Graphical Cost Representation. The cost parameters, the total demand, and the criterion of cost minimization are used to determine the economic order quantity, Q^*. Figure 2 shows the behavior or several cost functions as the order size is varied.

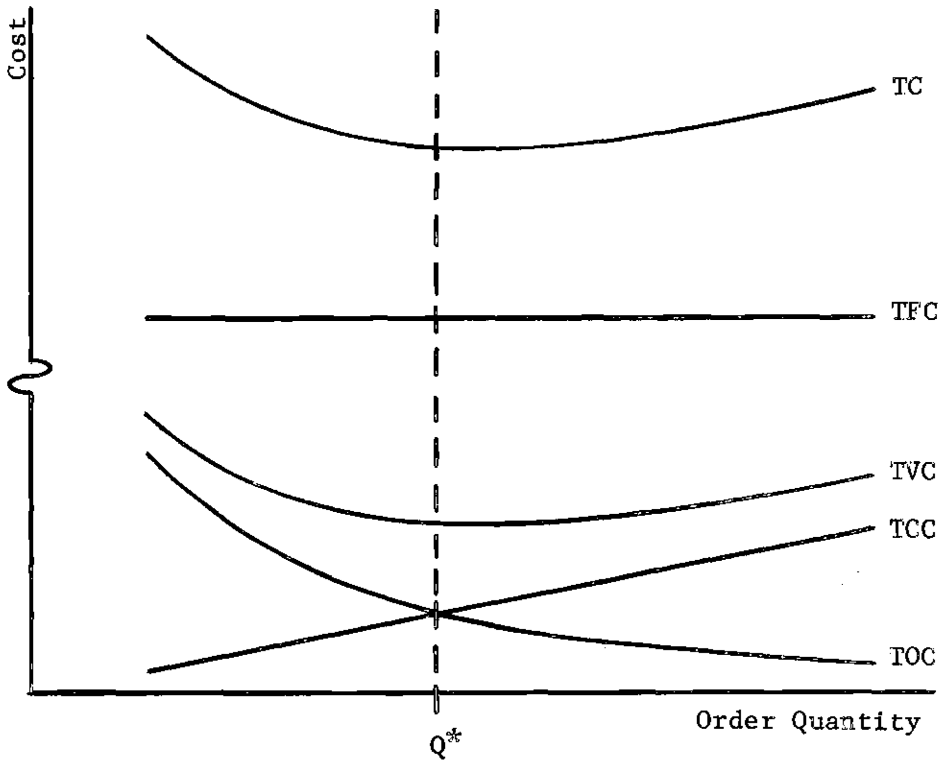

Figure 2. Balancing of Opposing Costs: Harris' Model.

The total variable cost function, TVC, is composed of the total carrying cost function, TCC, and the total ordering cost function, TOC. As the order quantity increases, the number of orders necessary to meet the demand and, therefore, the total ordering cost decreases. As the order quantity is increased, however, the total carrying cost increases.

The total cost function, TC, is composed of the total variable cost function and the total fixed cost function, TFC. Since the unit price and the total demand are both constant, the total fixed cost affects only the magnitude of the total cost function, not its shape. The total variable cost function affects both the magnitude and the shape of the total cost function. Figure 2 shows that the economic order quantity occurs at the minimum point of the total cost curve and, consistent with the preceding discussion, at the similar point of the total variable cost curve. It will therefore be possible to confine the discussion to the use of the total variable cost function for determining the economic order quantity, Q^*.

Figure 2 further shows that Q^* occurs when the total ordering cost function and the total carrying cost function intersect, and are equal. This gives rise to the concept of "balancing opposing costs."

Discussion of Time Periods. In using the Harris model, demand will usually be given in units per time period. It is expected that the demand will be constant over several of these periods. The time for which the demand is expected to remain constant will be called the forecast period. The time period in which the actual demand is usually expressed will be called the decision period. The need for such a distinction between time periods will become more apparent when other demand-patterns are considered.

Mathematical Derivation. Since it has been shown that the total variable cost curve has a minimum, at which Q^* occurs, the methods of calculus may be used to derive the formula for the economic order quantity from the equation of total variable cost. The parameters of the total variable cost equation, and their associated units for this discussion, will be given by the following notation:

t = decision period;

T = forecast period (decision periods per forecast period);

d = decision period demand (units per decision period);

D = forecast period demand (units per forecast period);

S = ordering cost (dollars per order);

U = unit cost or price (dollars per unit);

i = decision period carrying cost (percentage per unit per decision period);

I = forecast period carrying cost (percentage per unit per forecast period);

Q = order quantity (units);

(tvc) = decision period total variable cost; and

(TVC) = forecast period total variable cost.

Since d and i are assumed to be constant, their forecast period counterparts, D and I, will simply be T/t times greater.

The total variable cost equation may be written using either the parameters of the decision period or those of the forecast period. Care must be taken to insure that the units are consistent. Equation 1 is the total variable cost for a decision period, (tvc), and equation 2 is the total variable cost for a forecast period, (TVC).

$$(tvc) = \frac{d}{Q}(S) + \frac{Q}{2}(i)(U) \tag{1}$$

$$(TVC) = \frac{D}{Q}(s) + \frac{Q}{2}(I)(U) \tag{2}$$

Differentiating each equation with respect to Q; setting the result equal to zero; solving for Q; and calling this order quantity the economic order quantity, Q^*, the following equations are obtained (See Appendix, pages 60 and 61.):

$$Q^* = \sqrt{\frac{2dS}{iU}} \tag{3}$$

and

$$Q^* = \sqrt{\frac{2DS}{IU}} \tag{4}$$

Equation 3 is derived from equation 1 and equation 4 from equation 2. Proof that Q^* is actually the order quantity for which (tvc) and (TVC) are a minimum is obtained by showing that the second derivatives of (tvc) and TVC are positive. (See Appendix, pages 60 and 61.)

Equations 3 and 4 will actually yield the same economic order quantity, providing the units are consistent. To show this, it is only necessary to remember that D is equal to (T/t) times d and I is equal to (T/t) times i. Substituting these relationships in equation 4, and simplifying, equation 3 can be obtained.

The economic order quantity has been found to be function of the cost parameters and the square root of demand. Since the cost parameters S, U, i, and I are constants, the equations for Q^* may be re-written as follows:

$$Q^* = \sqrt{\frac{2S}{iU}} \cdot \sqrt{d} = k \cdot \sqrt{d} \tag{3a}$$

and

$$Q^* = \sqrt{\frac{2S}{IU}} \cdot \sqrt{D} = K \cdot \sqrt{D} \tag{4a}$$

Lower case k will represent the cost parameter relationship associated with Q^* expressed as a function of the decision period demand and upper case K will denote the relationship when Q^* is expressed as a function of the forecast period demand.

Once the economic order quantity has been found, the other parameters of the inventory-behavior-pattern are determined. The economic number of orders per decision period, n^*, and the economic number of orders per forecast period, N^*, may be found by dividing the respective period demands by Q^*. The economic inventory cycle time, t_c^*, may be found by taking the reciprocal of either n^* or N^*.

Modifications of Harris' Model

Discussion of Presentation

Selected models, based upon modification or elimination of one or more of Harris' basic assumptions, will be presented in the following sections. These models will be investigated in Chapter IV to determine the ease or feasibility with which they could be adapted to cover the case of known demand-with-trend. The nomenclature of the models was chosen to give clarity of recall when their adaptations to cover known demand-with-trend are discussed in Chapter IV.

For each model, Harris' assumptions which have been modified will be discussed; the situations for which the model was designed will be presented; additional parameters will be given whenever necessary; and, finally, the model itself will be presented.

A summary of the economic order quantity models in this chapter will be presented in the last section.

<u>Production EOQ Model</u>

The only assumption of Harris that is modified is that the receipt into stock occurs instantaneously. In this model, it is assumed that the inventory units are not ordered, but are produced within the firm. Consequently, they will be placed in inventory at some finite rate. This rate is called the production rate, P. The order cost, S, must now represent the cost of preparing the machines for production of the item, or the set-up cost.

The economic order quantity determined by this model may be written as follows:

$$Q^* = \sqrt{\frac{2SPd}{iU(P - d)}} = \sqrt{\frac{2SP}{iU}} \cdot \sqrt{\frac{d}{P - d}} = k_P \cdot \sqrt{\frac{d}{P - d}} \qquad (5)$$

The production model actually includes the Harris model as a special case. In the Harris model, the production rate is assumed to be infinite. As the production rate approaches infinity, the ratio $P/(P - d)$ approaches unity and the formula for Q^* approaches that of the Harris model.

<u>Non-Inventory EOQ Model</u>

This model assumes that the costs of a shortage and of ordering are finite, but that the carrying cost is infinite. This situation will occur when no storage space for inventory can be provided.

The economic order quantity for the non-inventory model is written as follows, with π in dollars per unit shortage per period:

$$Q^* = \sqrt{\frac{2dS}{\pi}} = \sqrt{\frac{2S}{\pi}} \cdot \sqrt{d} = (k_\pi)_2 \cdot \sqrt{d} \qquad (6)$$

Quantity Discount EOQ Models

For many items in an inventory, the unit price will not be constant. If it is purchased from an outside supplier, there may be several prices or discounts given for ranges of order sizes. Transportation and other similar costs may also be reduced as the order quantity is increased. Rather than developing a model to take into consideration quantity discounts, most references give a set of decision rules.

For the case of two unit prices, with P_1 greater than P_2, the decision rules may be summarized as follows:

(1) compute $Q^*_{(P_2)}$, using an appropriate model;

(2) if this quantity is greater than or equal to the quantity necessary to obtain P_2, order $Q^*_{(P_2)}$;

(3) if the quantity computed is less than the order quantity necessary to obtain P_2, then:

  (a) compute the total cost associated with ordering the minimum quantity needed to get P_2;

  (b) compute $Q^*_{(P_1)}$ and its associated total cost; and

  (c) comparing the two total costs, order that quantity yielding the lowest total cost.

For more than two unit prices, the decision rules would be similar to those illustrated above. References to models incorporating a specific number of quantity discounts will be found in the summary of models at the end of this chapter.

Summary of the Economic Order Quantity Models Presented

A summary of the models presented in this chapter is shown in Table 1. The table was prepared to give ready access to the assumptions of each model and to show selected sources where further discussions of the models may be found. The phrase "economic order quantity" has been abbreviated to EOQ to conserve space.

Table 1. Economic Order Quantity Models.

Model	Cost Parameter Assumptions Carrying Cost	 Ordering Cost	 Shortage Cost	References
Harris EOQ Model	Constant	Constant	Infinite	(2, p. 79)
"	"	"	"	(15, p. 14)
"	"	"	"	(16, p. 9)
"	"	"	"	(18, p. 202)
Production EOQ Model	Constant	Constant	Infinite	(16, p. 10)
"	"	"	"	(19, p. 342)
"	"	"	"	(20, p. 55)
Non-Inventory EOQ Model	Infinite	Constant	Constant	(19, p. 340)
Quantity Discount EOQ Model	Constant	Constant	Infinite	(18, p. 238)
"	"	"	"	(2, p. 84)
"	"	"	"	(9, p. 34)

CHAPTER III

DEVELOPMENT OF THE TREND EOQ MODEL

In this chapter, a method for determining the basic economic order quantities for Class II (known, with trend) demands will be developed. A mathematical formulation of the problem will be presented. Exact solution of the formulation is not practicable, so an approximate solution will be presented. This solution will take the form of a multiple regression equation for determining the correction in order quantity necessitated by the trend in demand.

Assumptions and Limitations

The importance of distinguishing between a decision period and a forecast period, and between their parameters, becomes clear with the introduction of one of the assumptions necessary to the development of the Trend EOQ Model. This assumption is:

> if the forecast period demand is known, with a trend, the decision period must be chosen so that its demand will be constant within any given decision period.

In other words, if the forecast period demand is a Class II demand, the decision period demand must be a Class I demand. The effect of this assumption is to approximate a linear function of demand using a step-function of demand, as is shown in Figure 3. The smaller the decision period, the better the approximation and the greater the validity of the assumption. Similarly, the smaller the trend, the better the approximation.

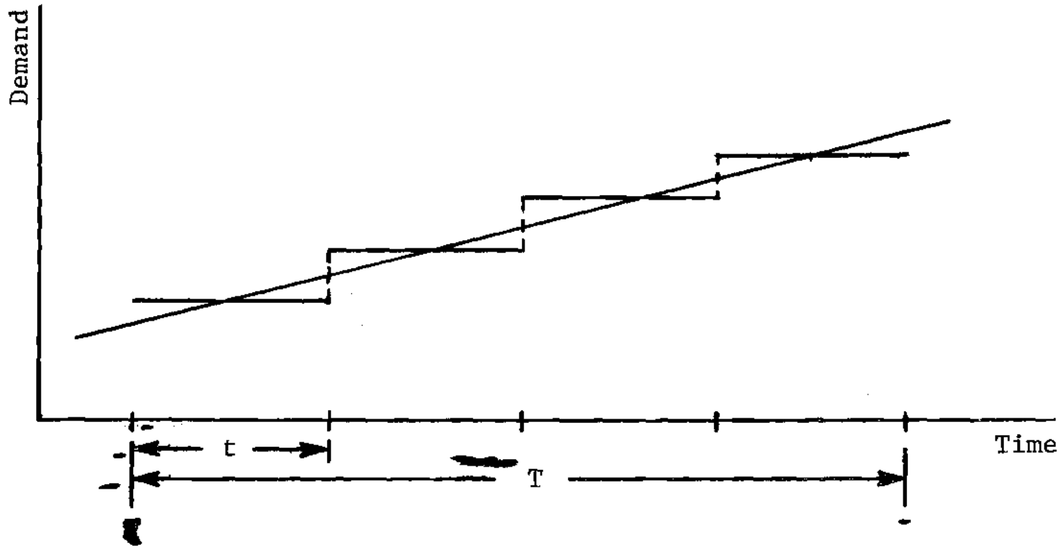

Figure 3. The Effect of Assuming a Constant Decision Period Demand When the Forecast Period Demand is Known-With-Trend.

The known demand-with-trend will be given in the form of a linear equation for the decision period demand as a function of time, trend, and a basic demand quantity. This form for the Class II demand leads to the desired form of the equation for the trend economic order quantity.

It is desired that the economic order quantity formula for known demand-with-trend be developed in the form of a linear function of time, a basic order quantity, and a correction factor necessitated by the trend in demand.

Mathematical Formulation

In accordance with the preceding section, the decision period demand for a Class II demand, d_t, will be given by the following equation:

$$d_t = B + At \tag{7}$$

B represents the basic demand (i.e., demand in the zero-th decision period), A represents the trend in demand, and t again denotes the decision period.

Denoting the cumulative demand through period t by D_t and the forecast period (T, as before) demand by D_T, their equations may be written:

$$D_t = Bt + \sum_{j=1}^{t} Aj = Bt + A \cdot \sum_{j=1}^{t} j \tag{8}$$

and

$$D_T = BT + \sum_{j=1}^{T} Aj = BT + A \cdot \sum_{j=1}^{T} j \tag{9}$$

The case of known and constant demand, for which the Harris Model can be applied, is then actually a special case of known demand-with-trend, in that the trend, A, is zero.

The basic economic order quantity, for known demand-with-trend, $Q_t'^*$, may be derived using the method presented in the preceding chapter, by substituting d_t and D_T in equations 3 and 4 for d and D. The equations for $Q_t'^*$ are:

$$Q_t'^* = k \cdot \sqrt{d_t} = k \cdot \sqrt{B + At} \tag{10}$$

and

$$Q_F'^* = K \cdot \sqrt{D_T} \quad K \cdot \sqrt{BT + A \cdot \sum_{j=1}^{T} j} \tag{11}$$

As previously stated, it is desired that the equation for $Q_t'^*$ be a linear function of time, a basic order quantity, and a correction factor necessitated by the trend in demand. If the basic order quantity is defined as the economic order quantity for the zero-th decision period, and the trend correction factor is denoted by lower case m, the desired form of $Q_t'^*$ may be written as:

$$Q_t'^* = (k \cdot \sqrt{B}) + mt \tag{12}$$

Transformation of either equation 10 or equation 11 to the desired form for $Q_t'^*$ appears to be impractical, if not also impossible.

One alternative to requiring the form of equation 12 would be to evaluate equations 10 and 11. One of these two evaluations would not yield a true economic order quantity, however, and the other would be required for each of the T decision periods.

Evaluation of the square root term in equation 11 would simply give an order quantity based upon a uniform or constant decision period demand of $(1/T) \cdot D_T$ throughout the forecast period. This, of course, would not be the trend economic order quantity. Determining the trend economic order quantity by use of equation 10 would require an evaluation for each decision period. While this would be a valid method, it involves many more calculations than would the use of equation 12.

The only unknown parameter in equation 12 is the trend correction factor, m. The value of m must be determined consistent with the cost minimization criterion. The decision period total variable cost equation, designated (tvc)' for known demand-with-trend, may be written as:

$$(tvc)' = \frac{(B + At)(S)}{(k \cdot \sqrt{B} + mt)} + \frac{(k \cdot \sqrt{B} + mt)(iU)}{2} \qquad (13)$$

To determine the economic trend correction factor, m_t^*, it is necessary to differentiate equation 13 with respect to m; set the result equal to zero; solve for m; and call the resulting quantity m_t^*. (See Appendix, page 62.) The equation for m_t^* is found to be:

$$m_t^* = k \cdot \frac{(\sqrt{B + At} - \sqrt{B})}{t} \qquad (14)$$

If m_t^* had been found to be a function of only the constant parameters B, A, and k, the desired form of $Q_t'^*$ could readily have been determined. The use of m_t^* in equation 12 would still require many evaluations of the square root terms during the duration of the forecast period.

The following section investigates the possibility of determining a constant-approximation to m_t^*, consistent with the criterion of cost minimization, in order to reduce the number of calculations required to establish economic ordering quantities for the duration of the forecast period.

A Constant-Approximation of the Trend Correction Factor

Notation and Rounding Procedure

Since most practical situations require that the order quantity be a whole number of units, the constant-approximations to m_t^* will be limited to integer values. M will denote an integer-valued, constant-approximation to m_t^*. M^* will denote that value of M, among all the M's, which gives the lowest forecast period total variable cost (i.e., M^* is the economic M).

The order quantity determined by substituting M in equation 12 will be denoted by Q'_M and, similarly, by using M^* in equation 12 will be given by Q'_{M^*}.

To ensure that Q'_M and Q'_{M^*} are actually whole numbers, it will be necessary to round-off any fractional portions of the basic economic order quantity, $(k \cdot \sqrt{B})$. The following rule or procedure will be used for rounding-off numbers in this investigation.

> if the integer to the right of the decimal point (or the position which is to be rounded) is five or greater, increase the integer to the left of the decimal point by one and drop the fractional portion; if it is less than five simply drop the fractional portion.

Two Order Quantity Policies for Known Demand-With-Trend

At this point, it would be useful to illustrate two economic order quantity policies based upon Q'_{t^*} and Q'_{M^*}. The following parameter values will be used in the illustration:

(1) $d_t = 500 + 50t$;

(2) $T = 12$ (i.e., $t = 1, 2, \ldots, 12$);

(3) $S = \$10.00$ per order;

(4) $i = 2$ percent per unit per decision period; and

(5) $U = \$2.00$ per unit.

The value of k would then be:

$$k = \sqrt{\frac{2S}{iU}} = \sqrt{\frac{(2)(10)}{(0.02)(2)}} = \sqrt{500} = 22.36068 \qquad (15)$$

The policy based upon $Q_t'^*$ would give the lowest-possible, or optimum, forecast period total variable cost, which will be denoted $(TVC)'^*$. The values of $Q_t'^*$ and of $n_t'^*$, the optimal number of orders, for each decision period of the example are presented in Table 2. (See Appendix, page 62.) The forecast period total variable cost, $(TVC)'^*$, for this policy is \$306.56. (See Appendix, page 63).

Table 2. Optimal Economic Order Quantity Policy
For: $d_t = 500 + 50t$

t	d_t	$Q_t'^*$	$n_t'^*$	m_t^*
1	550	524.404	1.0488	24.404
2	600	547.723	1.0954	23.861
3	650	570.088	1.1402	23.363
4	700	591.608	1.1832	22.902
5	750	612.372	1.2247	22.474
6	800	632.456	1.2649	22.076
7	850	651.920	1.3038	21.703
8	900	670.820	1.3416	21.353
9	950	689.202	1.3784	21.022
10	1,000	707.107	1.4142	20.711
11	1,050	724.569	1.4491	20.415
12	1,100	741.620	1.4832	20.135
Totals	9,900	7,663.889	15.3275	------

The values of m_t^* are also given in Table 2. It can be seen that m_t^* varies from 24.404 to 20.135, throughout the forecast period. It appears logical that the value of M^* would lie within the range of variation of m_t^*, and the values of M will be selected from this range. This will give M the following values: 24, 23, 22, 21, and 20. For a more significant illustration of the fact that M^* really exists (i.e., there is a constant approximation which has the lowest forecast period total variable cost of all the constant-approximations), the following additional values for M will be considered: 5, 10, 15, 25, 30, 35, and 40. Table 3 shows the selected values of M and the forecast period total variable costs, $(TVC)'_M$, of each policy resulting from substituting the values of M in the equation for Q'_M. (See Appendix, pages 64-68.) The basic economic order quantity, $(k \cdot \sqrt{B})$ is 500 units, for this example.

Table 3. Order Quantity Policy Costs For: $Q'_M = 500 + Mt$.

Value of M	Forecast Period Total Variable Cost
5	$312.65
10	309.22
15	307.31
18	306.75
19	306.65
20	306.59
21	306.57
22	306.59
23	306.64
24	306.73
25	306.85
30	307.92
35	309.66
40	311.97

Figure 4 is a graphical representation of Table 3 and the magnitude of $(TVC)'^*$. Using Table 3 or Figure 4, it can be shown that the minimum, or optimum, $(TVC)'_M$ is equal to $(TVC)'_{21}$. Thus, M^* is equal to 21. $(TVC)'_{21}$ is greater than $(TVC)'^*$ by only one cent, indicating that an approximation of m^*_t by an integer-valued constant is practicable.

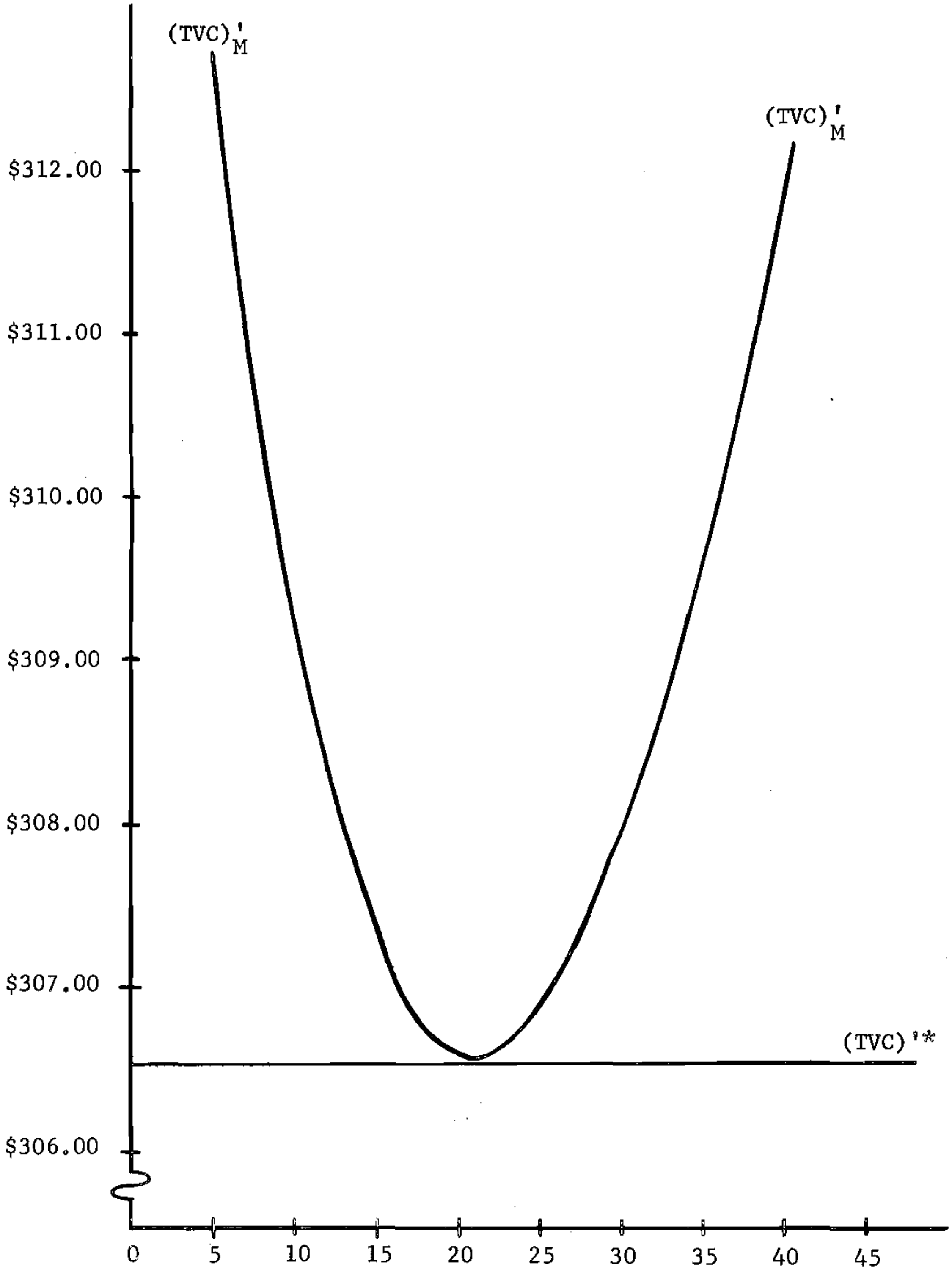

Figure 4. $(TVC)'_M$ For: $Q'_M = 500 + Mt$.

Formulation of a Direct Solution for M^*

From Figure 4, it appears that the value of M^* could be determined, using calculus, directly from the equation for $(TVC)'_M$. This would be possible, but not very practicable. For the example just presented, with a forecast period of twelve decision periods, determination of M^* directly would require the solution of a 24-th degree eqution. (See Appendix, page 69.) For other forecast periods, the degree of the equation to be solved for M^* would be 2T.

Proof that M^* exists and is a minimum point on the $(TVC)'_M$ curve is given by showing that the second derivative of $(TVC)'_M$ with respect to M is positive. (See Appendix, page 70.)

Discussion of Deviations in Total Variable Costs

For the case of known demand-with-trend, and under the previous assumptions, the use of $Q_t'^*$ in an economic order quantity policy will give the lowest-possible, or optimum, forecast period total variable cost, $(TVC)'^*$. Two other minimum-cost policies may be developed.

One of these policies, the policy based upon $Q_M'^*$, will give the lowest-possible forecast period total variable cost consistent with the requirement that the trend correction factor be an integer-valued constant. The forecast period total variable cost for this policy has been denoted $(TVC)'_{M^*}$.

The second minimum-cost policy that may be developed is based upon the false assumption that there is no trend in the forecast period demand. Under this assumption, the order quantity would be a constant throughout the forecast period and be given by equation 4. The forecast period total variable cost associated with this policy will be denoted

$(TVC)'_F$. The prime superscript indicates that the demand is actually known, with trend, and the subscript F indicates that the calculation of the order quantity assumes falsely that the demand is known and constant.

Comparisons among the forecast period total variable costs of the three economic order quantity policies mentioned above will measure:

(1) the effect of neglecting the fact that there is actually a trend in demand.

(2) the effect of approximating the optimum economic order quantity policy by a policy which increases the order quantity by a fixed, integer-valued amount each successive decision period.

The following variables will be used in comparing the forecast period total variable costs of the three economic order quantity policies:

$$\text{Maximum Loss} = (TVC)'_F - (TVC)'^* \tag{16}$$

$$\text{Minimum Loss} = (TVC)'_{M^*} - (TVC)'^* \tag{17}$$

The preceding variables will also be expressed as percentages of the optimum forecast period total variable cost, $(TVC)'^*$, for a more general basis of comparison.

The order quantity associated with $(TVC)'_F$ will be denoted Q'_F. For the preceding example, Q'_F would be 642.26 units and $(TVC)'_F$ would be \$308.28. (See Appendix, page 63.)

The following cost comparisons for the preceding examples may now be presented:

$$\text{Maximum Loss} = \$308.28 - \$306.56 = \$1.72 \tag{18}$$

$$\text{Maximum Loss (\%)} = (\$1.72/\$306.56) \cdot (100\%) = 0.56\%, \quad (19)$$

$$\text{Minimum Loss} = \$306.57 - \$306.56 = \$0.01, \quad (20)$$

and

$$\text{Minimum Loss (\%)} = (\$0.01/\$306.56) \cdot (100\%) = 0.00\% \quad (21)$$

Inventory-Behavior-Patterns Under Known Demand-With-Trend

Three different order quantity policies have been used for the case of known demand, with trend. Each policy can be considered optimum, or economic, under its own assumptions. (e.g., if trend is neglected, Q'_F will be the true optimum order quantity policy; etc,) The inventory fluctuations under these policies are represented in Figures 5, 6, and 7. The data used in determining the inventory-behavior-patterns was taken from the preceding example. The axes' scales are not given, but are the same for each figure.

Figure 5 shows a constant order quantity and an increased frequency of ordering as t increases. Figure 6 shows a variable-increasing order quantity as t increases, since Q'^*_t is based upon m^*_t. Finally, figure 7 shows the uniform-increasing order quantity based upon the constant M^*. Since it will sometimes be necessary to order twice during a decision period, the order quantities in figures 6 and 7 may not always increase with each order.

The order quantities determined by Q'^*_t and Q'_{M^*} are less than Q'_F at the beginning of the forecast period and greater than Q'_F at its end.

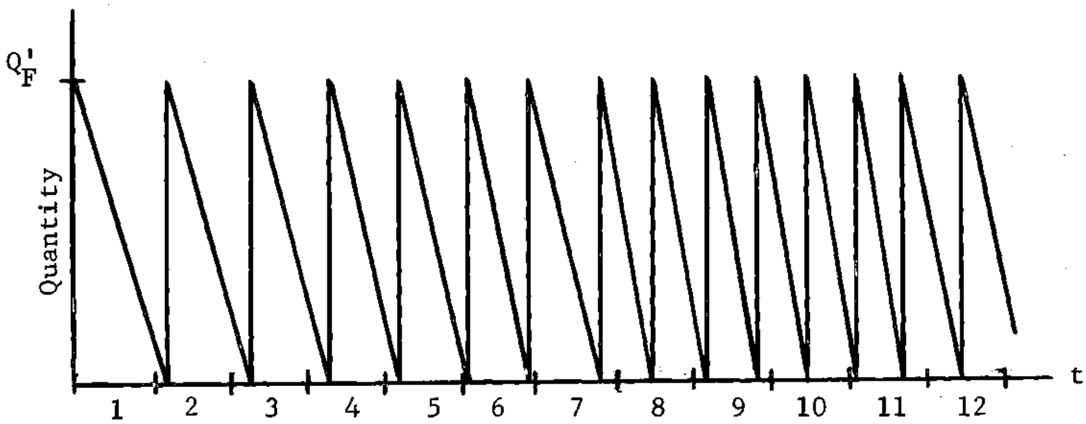

Figure 5. Inventory-Behavior-Pattern: Q'_F.

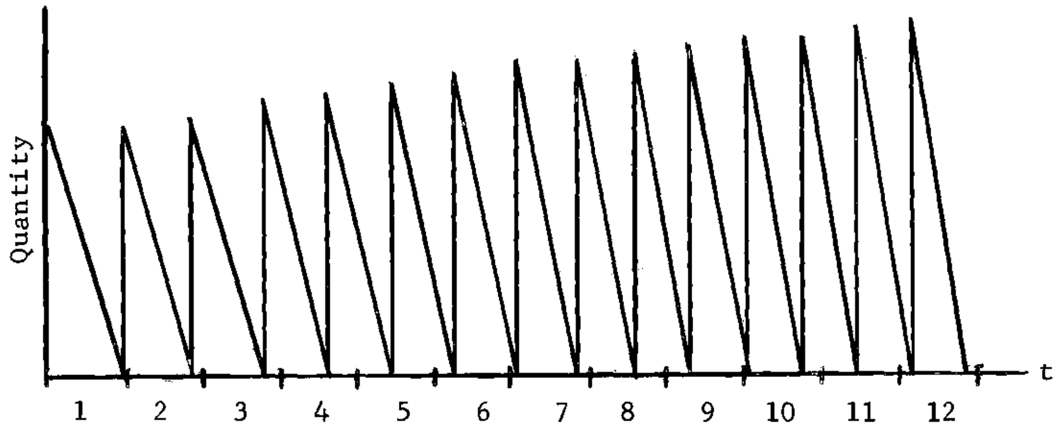

Figure 6. Inventory-Behavior-Pattern: Q'^*_t.

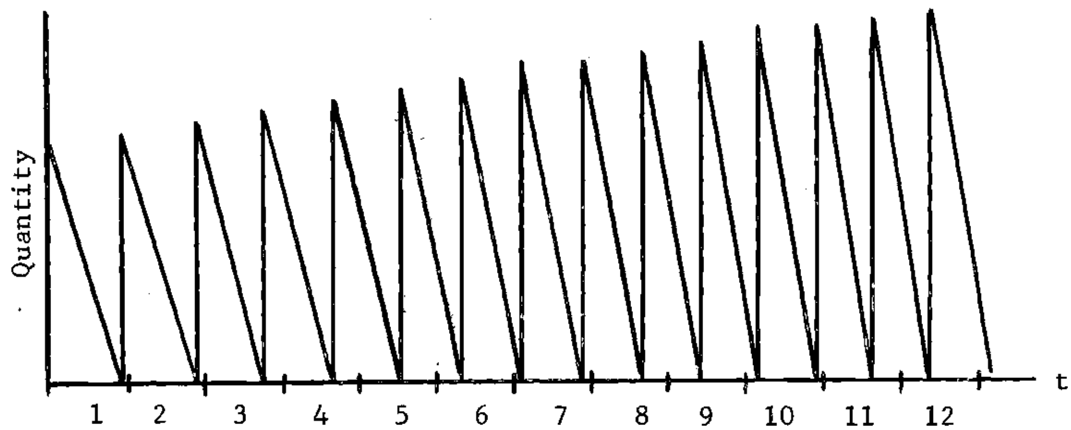

Figure 7. Inventory-Behavior-Pattern: Q'_{M^*}.

A Multiple Regression Equation for Predicting M^*

The Desirability of Predicting M^*

The determination of M^* and its order quantity policy parameter, Q'_{M^*}, in the preceding example, was obviously more time-consuming than the development of the $Q_t'^*$ order quantity policy. The main reason for desiring the trend economic order quantity in the form of a linear equation was to reduce the complexity of the required calculations. (The use of Q'_{M^*}, once M^* has been determined, requires only the addition of M^* units to the preceding period's order quantity.) If a simple predictor of M^* can be developed, the complexity and time of the calculations necessary for use of Q'_{M^*} will be favorably reduced, and the model will be practicable. Errors of a magnitude of a few units in the predicted values of M^* can be tolerated, because of the relative flatness of the $(TVC)'_M$ curve near its minimum. (i.e., $(TVC)'_M$ is relatively insensitive to changes in M within several units of M^*.)

Foundation for the Multiple Regression Analyses

Independent Variables. The formula developed for m_t^* (equation 14) and the attempt at solving the equation of the $(TVC)'_M$ curve directly for M^* suggest the independent variables which should be used in the prediction of M^*. These variables, common to both equations, are: B, the basic demand; A, the trend in demand; k, the square root of twice the cost ratio; and T, the forecast period. Two slight modifications of these variables should be made to allow further generalization of the regression analyses.

The trend in demand, A, should be converted into a percentage of the basic demand. The variable k should be replaced by its corresponding cost ratio, (S/iU).

Levels of the Independent Variables. Each variable was presented at three levels. These three levels were selected at equal intervals. All possible combinations of the variables at each level were investigated, giving eighty-one, (3^4), observations upon which to base the regression analysis. The levels of the variables were:

B_1 = 500 units per decision period,

B_2 = 3,000 units per decision period, and

B_3 = 5,500 units per decision period;

A_1 = 5% of basic demand per decision period,

A_2 = 10% of basic demand per decision period, and

A_3 = 15% of basic demand per decision period;

$(S/iU)_1$ = 250,

$(S/iU)_2$ = 200, and

$(S/iu)_3$ = 150

and

T_1 = 12 decision periods,

T_2 = 24 decision periods, and

T_3 = 36 decision periods.

The Dependent Variable M^*. The value of M^* for each of the 81 combinations of the independent variables and their levels was determined by the method of the preceding example. (It should be noted that the previous example represents the observation for: $B_1A_2(S/iU)_1T_1$.) The values of $M^* - 1$, M^*, and $M^* + 1$ are given in the Appendix, along with their associated forecast period total variable costs. $(TVC)'_{M^*-1}$ and $(TVC)'_{M^*+1}$ are each greater than $(TVC)'_{M^*}$, assuring that the value of M^* shown actually yields a minimum forecast period total variable cost policy. (See Appendix, pages 71 and 72.) A summary of the values of M^* for the 81 observations is given in Table 4.

The Regression Analyses

Two equations for predicting M^* from the independent variables were investigated: a linear equation and a logarithmic-linear equation. Least squares regression equations were developed for both relationships, using the Burroughs 220 Data Processing System at the Rich Electronic Computer Center. (21)

If the predicted value of M^* is denoted by M^*_P, the linear regression equation, based upon the 81 observations, may be written:

$$M^*_P = -29.422221 + 0.0077333336(B) + 3.1777777(A)$$
$$0.096666666(S/iU) - 0.30401234(T) \qquad (22)$$

The standard error and the multiple correlation coefficient, R, for the linear regression equation were found to be:

$$(\text{Standard Error})_{\text{Linear}} = 6.7354874 \qquad (23)$$

and

$$(R)_{\text{Linear}} = 0.95231420 \qquad (24)$$

Table 4. Values of M^*

		B_1			B_2			B_3		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$(S/iU)_1$	T_1	11	21	30	28	51	73	38	70	99
	T_2	11	19	26	26	46	64	35	63	86
	T_3	10	17	24	24	43	58	33	58	79
$(S/iU)_2$	T_1	10	19	27	25	46	65	34	62	88
	T_2	9	17	23	23	41	57	31	56	77
	T_3	9	16	21	22	38	52	30	52	70
$(S/iU)_3$	T_1	9	16	23	21	40	56	29	54	76
	T_2	8	15	20	20	36	49	27	48	67
	T_3	8	13	18	19	33	45	26	45	61

The linear-logarithmic equation was developed from the following expression:

$$M_P^* = a \cdot (B)^{b_1} \cdot (A)^{b_2} \cdot (S/iU)^{b_3} \cdot (T)^{b_4} \qquad (25)$$

The constant, (a), and the exponents, (b_1, b_2, b_3, and b_4), are to be determined by the regression analysis.

Equation 29 may be converted to a linear form by taking the logarithm of each side, thusly:

$$\log(M_P^*) = \log(a) + b_1 \cdot \log(B) + b_2 \cdot \log(A) + b_3 \cdot \log(S/iU) + b_4 \cdot \log(T) \qquad (26)$$

Common or Briggsian logarithms were used in developing the regression equation. The linear-logarithmic regression equation, based upon the 81 observations, was found to be:

$$\log(M_P^*) = -1.90172244 + 0.50055172\,[\log(B)] + 0.82941016\,[\log(A)] + 0.50657877\,[\log(S/iU)] - 0.16170453\,[\log(T)] \qquad (27)$$

The standard error and the multiple correlation coefficient for the linear-logarithmic regression equation were:

$$(\text{Standard Error})_{\text{Linear-Log}} = 0.010201670 \qquad (28)$$

and

$$(R)_{\text{Linear-Log}} = 0.99933948 \qquad (29)$$

The Final Equation

The multiple correlation coefficient of the linear-logarithmic regression equation indicates a very high degree of correlation between the values of the independent variables and the value of M^*. (Unity indicates perfect correlation.) Since this is higher than the multiple correlation coefficient of the linear regression equation, a linear-logarithmic expression will be selected as the predictor of M^*. For the final expression, the regression coefficients of equation 31 will be rounded to five decimal places. This gives the following equation for M^*_P:

$$\log(M^*_P) = -1.90172 + 0.50055\,[\log(B)] + 0.82941\,[\log(A)] + 0.50658\,[\log(S/100)] - 0.16170\,[\log(T)] \qquad (30)$$

The values of M^*_P determined by equation 30 must be rounded to integer values, in accordance with the restriction placed upon the values of M^*.

Values of M^*_P have been determined for each of the 81 observations, to investigate the accuracy of equation 30 in predicting the integer-values of M^*. (See Appendix, pages 73 and 74.) Table 5 gives a summary of M^*_P, the predicted integer values of M^*, for the 81 observations.

The deviation, or difference, between the predicted and actual integer-values of M^* is defined as:

$$(DEV) = M^*_P - M^* \qquad (31)$$

For the 81 observations, the range of deviations was: -2, -1, 0, +1, and +2. (See Appendix, pages and .) The fact that $(TVC)'_M$ is

Table 5. Integer Values of M_P^*

		B_1			B_2			B_3		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$(S/iU)_1$	T_1	12	21	29	29	51	72	39	69	97
	T_2	10	19	26	26	46	64	35	62	87
	T_3	10	17	24	24	43	60	33	58	81
$(S/iU)_2$	T_1	10	19	26	26	46	64	35	62	87
	T_2	9	17	23	23	41	57	31	55	72
	T_3	9	16	22	22	38	53	29	52	72
$(S/iU)_3$	T_1	9	16	23	22	39	55	30	53	75
	T_2	8	14	20	20	35	49	27	48	67
	T_3	8	13	19	19	33	46	25	45	63

relatively insensitive to changes in M within several units of M^* indicates that equation 30 would be an acceptable equation for predicting the values of M^*. Further indication of the desirability of equation 30 for use in the prediction of M^* will be given in later discussions of the cost comparison variables.

For the preceding example (i.e., using the observation $B_1A_2(S/iU)_1T_1)$, the values of M_P^* and (DEV) were found to be:

$$M_P^* = 21 \qquad (32)$$

$$(DEV) = M_P^* - M^* = 21 - 21 = 0 \qquad (33)$$

Cost Comparisons

Two variables, (Maximum Loss) and (Minimum Loss), were presented previously for use in comparing the forecast period total variable costs of economic order quantity policies that are based upon $Q_t'^*$, Q_F', and Q_{M^*}'. (See pages 27-29.) One additional variable will be necessary to measure the effect, upon forecast period total variable costs, of using equation 30 to predict values of M^* (i.e., M_P^*). This variable will be defined as follows:

$$\text{Expected Loss} = (TVC)'_{M_P^*} - (TVC)'^* \qquad (34)$$

This variable, (Expected Loss), will also be expressed as a percentage of $(TVC)'^*$, for a more general basis of comparisons to be drawn between the eighty-one observations.

Values of $(TVC)'^*$, $(TVC)_F'$, $(TVC)_{M^*}'$, and $(TVC)'_{M_P^*}$ were determined for each of the eighty-one observations, to investigate the cost comparison variables. (See Appendix, pages 76 and 77.) The actual and percentage values of the cost comparison variables were then determined for each of the eighty-one observations. (See Appendix, pages 78 and

79.) The percentage values are shown in Table 6.

The first entry for each observation represents the value of the (Maximum Loss) expressed as a percentage of $(TVC)'^*$. It can be seen that the magnitude of the basic demand, B, and of the cost ratio, (S/iU), apparently has no effect upon the magnitude of the percentage (Maximum Loss). The percentage trend in demand, A, and the length of the forecast period, T, however, appear to have a significant effect upon the magnitude of the percentage (Maximum Loss). As either A or T increases, the percentage (Maximum Loss) also increases.

The second entry under each observation, in Table 6, represents the value of the (Minimum Loss) expressed as a percentage of $(TVC)'^*$. The preceding discussion of the effects of B, A, (S/iU), and T also applies to the percentage (Minimum Loss). The third entry, percentage (Expected Loss), generally follows the same cause/effect pattern as the preceding entries. Variations from this pattern are caused by errors in the prediction of M^* through the use of equation 30.

For the eighty-one combinations of B, A, (S/iU), and T that were investigated, the following ranges of the cost comparison variables were observed:

(1) (Maximum Loss) - from 0.21% to 2.45% of $(TVC)'^*$.

(2) (Minimum Loss) - from 0.00% to 0.11% of $(TVC)'^*$.

(3) (Expected Loss)- from 0.00% to 0.13% of $(TVC)'^*$.

The preceding discussions have shown that: (Maximum Loss) measures the effect of neglecting trend: (Minimum Loss) measures the effect of approximating the optimum, variable change in order quantity by a uniform, integer-valued change, M^*; and (Expected Loss) measures the

Table 6. Percentage Values of Cost Comparison Variables

		B_1			B_2			B_3			Cost Comparison Variable
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3	
$(S/1U)_1$	T_1	0.21	0.56	0.90	0.21	0.56	0.90	0.21	0.56	0.90	Maximum Loss (%)
		0.00	0.00	0.01	0.00	0.01	0.01	0.00	0.00	0.01	Minimum Loss (%)
		0.00	0.00	0.02	0.00	0.01	0.01	0.00	0.00	0.05	Expected Loss (%)
	T_2	0.59	1.27	1.80	0.59	1.27	1.80	0.59	1.27	1.80	Maximum Loss (%)
		0.01	0.02	0.05	0.00	0.02	0.05	0.00	0.02	0.05	Minimum Loss (%)
		0.01	0.02	0.05	0.00	0.02	0.05	0.00	0.02	0.05	Expected Loss (%)
	T_3	0.96	1.84	2.45	0.96	1.84	2.45	0.96	1.84	2.45	Maximum Loss (%)
		0.01	0.06	0.11	0.01	0.05	0.11	0.01	0.05	0.11	Minimum Loss (%)
		0.01	0.06	0.11	0.01	0.05	0.12	0.01	0.05	0.12	Expected Loss (%)
$(S/1U)_2$	T_1	0.21	0.57	0.90	0.22	0.56	0.90	0.21	0.56	0.90	Maximum Loss (%)
		0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	Minimum Loss (%)
		0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	Expected Loss (%)
	T_2	0.58	1.27	1.80	1.59	1.27	1.80	0.59	1.27	1.80	Maximum Loss (%)
		0.01	0.03	0.06	0.00	0.02	0.05	0.00	0.02	0.05	Minimum Loss (%)
		0.01	0.03	0.06	0.00	0.02	0.05	0.00	0.03	0.05	Expected Loss (%)
	T_3	0.96	1.84	2.45	0.96	1.84	2.45	0.96	1.84	2.45	Maximum Loss (%)
		0.01	0.06	0.11	0.01	0.05	0.11	0.01	0.05	0.11	Minimum Loss (%)
		0.01	0.06	0.13	0.01	0.05	0.12	0.01	0.05	0.12	Expected Loss (%)
$(S/1U)_3$	T_1	0.22	0.56	0.90	0.21	0.56	0.90	0.21	0.56	0.90	Maximum Loss (%)
		0.00	0.00	0.01	0.00	0.00	0.01	0.00	0.00	0.01	Minimum Loss (%)
		0.00	0.00	0.01	0.00	0.01	0.02	0.00	0.01	0.01	Expected Loss (%)
	T_2	0.59	1.27	1.80	0.59	1.27	1.80	0.59	1.27	1.80	Maximum Loss (%)
		0.01	0.03	0.05	0.00	0.02	0.05	0.00	0.02	0.05	Minimum Loss (%)
		0.01	0.03	0.05	0.00	0.03	0.05	0.00	0.02	0.05	Expected Loss (%)
	T_3	0.96	1.84	2.44	0.96	1.84	2.45	0.96	1.84	2.45	Maximum Loss (%)
		0.02	0.06	0.11	0.01	0.05	0.11	0.01	0.05	0.11	Minimum Loss (%)
		0.02	0.06	0.12	0.01	0.05	0.12	0.01	0.05	0.12	Expected Loss (%)

effect of predicting M^* by the use of equation 30. A positive, non-zero percentage (Loss) indicates that the forecast period total variable costs have increased from the optimum, $(TVC)'^*$.

The values of the cost comparison variables (Maximum Loss) and (Minimum Loss), for the continuing example, were presented in a preceding section. (See page 28.) The value of the cost comparison variable (Expected Loss) for the example is shown below:

$$\text{Expected Loss} = \$306.57 - \$306.56 = \$0.01 \qquad (35)$$

$$\text{Expected Loss (\%)} = (\$0.01/\$306.56) \cdot (100\%) = 0.00\% \qquad (36)$$

Effectiveness of the Model

The presentation of cost comparison variables in the preceding section was desinged to show the relative effects, upon forecast period total variable costs, of the four order quantity policies based upon $Q_t'^*$, Q_F', Q'_{M^*}, and $Q'_{M^*_P}$. Application of the Harris model, Q_F', when there is actually a trend in demand, will not give the lowest-possible forecast period total variable costs. Savings could be achieved through the use of any of the policies based upon $Q_t'^*$, Q'_{M^*}, or $Q'_{M^*_P}$. The first two of these three policies would require extensive calculations to determine the economic order quantities for each of the decision periods within the forecast period. The policy based upon $Q'_{M^*_P}$ will not always achieve the maximum reduction in forecast period total variable costs from $(TVC)_F'$, but will require far fewer calculations to determine the economic order quantities for each of the decision periods. To more easily evaluate the effectiveness of the economic order quantity model based upon $Q'_{M^*_P}$, a direct comparison will be drawn between it and the basic model developed by Harris. (Hereafter, the term "model", when used alone, will refer to the use of $Q'_{M^*_P}$.)

The effectiveness of the model will measure the extent to which the model can achieve the savings represented by the difference between $(TVC)'_F$ and $(TVC)'^*$ (i.e., the maximum or optimum savings). The following variable will be used in this measurement:

$$\text{Model Effectiveness} = \left[\frac{(\text{Maximum Loss}) - (\text{Expected Loss})}{(\text{Maximum Loss})}\right] \cdot (100\%) \tag{37}$$

(See Appendix, page 80.) The value of this variable was determined for each of the eighty-one observations, and the results are shown in Table 7. The model was found to be no less than ninety-four percent effective in achieving the maximum savings for the selected values of B, A, (S/iU), and T. The variables B and (S/iU) do not appear to have a significant effect upon (Model Effectiveness), while (Model Effectiveness) appears to decrease as either A or T increases.

The (Model Effectiveness) for the continuing example was found to be:

$$\text{Model Effectiveness} = \left[(\$1.72 - \$0.01)/(\$1.72)\right] \cdot (100\%) = 99\% \tag{38}$$

Table 7. Model Effectiveness (Percentage Values)

		B_1			B_2			B_3		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$(S/iU)_1$	T_1	98	99	98	99	99	99	99	99	94
	T_2	98	98	97	99	98	97	99	98	97
	T_3	99	97	95	99	97	95	99	97	95
$(S/iU)_2$	T_1	100	99	99	98	99	98	99	99	99
	T_2	98	98	97	99	98	97	99	98	97
	T_3	99	97	95	99	97	95	99	97	95
$(S/iU)_3$	T_1	100	99	99	100	99	98	99	99	98
	T_2	99	97	97	99	98	97	99	98	97
	T_3	98	96	95	99	97	95	99	97	95

Summary of the Model

The model presented in this chapter for determining the economic order quantities under the conditions of known demand-with-trend may be represented by

$$Q'_{M_P^*} = k \cdot \sqrt{B} + M_P^* \cdot t \tag{39}$$

in which the values of $k \cdot \sqrt{B}$ and M_P^* are integer-valued and constant. The value of M_P^* is to be determined through the use of

$$\log(M_P^*) = -1.90172 + 0.50055\,[\log(B)] + 0.82941\,[\log(A)] + 0.50658\,[\log(S/iU)] - 0.16170\,[\log(T)] \qquad (40)$$

where: B is the basic demand; A is the trend in demand, expressed as a percentage of B; (S/iU) is the ratio of cost parameters; and T is the length of the forecast period.

CHAPTER IV

TESTS AND ADAPTATIONS OF THE MODEL

In this chapter, the behavior of the model for the determination of economic order quantities under known demand-with-trend will be investigated for values of the independent variables differing from those used in the development of the model. Possible adaptations of the model, to include those modifications of the Harris model presented in Chapter II, will be discussed.

Tests of the Model

Five variables were presented in the preceding chapter for use in the evaluation of the economic order quantity model based upon $Q'_{M_P^*}$. These variables are: (DEV), (Maximum Loss), (Minimum Loss), (Expected Loss), and (Model Effectiveness).

A continuing example was used to illustrate the methods used in the development and application of the concepts and variables presented in this study. Since the values of B, A, (S/iU), and T used in the example were also used in the development of equation 30, for predicting the value of M^*, the example does not constitute an objective test of the model. This example, however, will be included in the following presentation of tests of the model, for the dual purposes of summarization and comparison.

Five additional tests of the model were performed. The tests were chosen to investigate the worth of the model under conditions involving values of the independent variables differing from those used in the development of equation 30. The specific conditions to be investigated by each of the tests are shown below:

Test #1 - summarization of the continuing example of $B_1A_2(S/iU)_1T_1$, and for comparison to the results of the other tests.

Test #2 - values of the independent variables B, A, (S/iU), and T near their magnitudes used in the development of equation 30.

Test #3 - values of the independent variables approximately midway between their values used in the development of equation 30.

Test #4 - values of A and T which approximate one-half the demand for a seasonal sales item and values of B and (S/iU) within the limits of their values used in the development of equation 30.

Test #5 - a value of T which would approximate a year's forecast, on a weekly basis, and corresponding values of the remaining independent variables.

Test #6 - values of B, A, and (S/iU) exceeding those values used in the development of equation 30 and a value of T to once again approximate one-half the forecast period of a seasonal sales item.

Table 8 shows, for the six Model Tests, the selected values of the independent variables; the values of the actual and predicted integer-valued constant-approximations; and the resulting values of the five dependent variables to be used in the evaluation of the model. (See Appendix, pages 81-85).

The range of deviation, (DEV), for the five additional tests was from zero (0) to minus-three (-3) units. As would be expected, the largest absolute deviation occurred in Model Text #6, for which the values of the independent variables exceeded those used in the development of the equation for M_P^*.

Table 8. Results of Tests of the Model

Variable	Model Test #1	Model Test #2	Model Test #3	Model Test #4	Model Test #5	Model Test #6
B (in units)	500	3,500	4,300	2,500	6,000	8,000
A (in % of B) (in units)	10% (50)	6% (210)	7% (301)	24% (600)	2.5% (150)	24% (1,920)
(S/iU)	250	240	225	170	160	300
T (in periods)	12	15	17	6	50	6
M^*_P	21	34	41	89	14	212
M^*	21	34	41	90	15	215
(DEV) (in units)	0	0	0	-1	-1	-3
Maximum Loss (%)	0.56%	0.39%	0.57%	0.64%	0.63%	0.64%
Minimum Loss (%)	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%
Expected Loss (%)	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%
Model Effectiveness	99%	99%	99%	99%	99%	99%

The range of (Maximum Loss) was from 0.39% to 0.64% of (TVC)'*. The range of (Minimum Loss) was from 0.00% to 0.01% of (TVC)'*. The range of (Expected Loss) was also from 0.00% to 0.01% of (TVC)'*.

Each of the five additional tests were found to have a (Model Effectiveness) of ninety-nine percent.

Model Tests #2 and #3, using values of the independent variables within the ranges considered in the development of equation 30, resulted in correct predictions of the desired values of M^*.

The preceding chapter indicated that the trend in demand, A, and the length of the forecast period, T, would have the greatest effect upon the cost comparison variables. Model Tests #4 and #5 were selected to investigate the value of (Model Effectiveness) as the values of the variables A and T, respectively, were increased to exceed those considered in the development of equation 30. The results of these Model Tests seem to indicate that equation 30 would be sufficiently accurate for predicting M^* for larger values of A and T than those that were considered in the eighty-one observations.

Model Test #6 increased the values of B, A, and (S/iU) above the maximum values used in developing equation 30. The value of T was increased to six, as in Model Test #4, to approximate one-half the forecast period for a seasonal sales item. No large reduction in (Model Effectiveness) was shown. This indicates that equation 30 could be used, without any decisive reduction in (Model Effectiveness), when the value of more than one independent variable exceeds those values used in the development of equation 30.

Considering the small values of (DEV) found among the eighty-one observations and five Model Tests, and the relative flatness of the $(TVC)'_M$ curve near M^*, equation 30 would be sufficiently accurate for practical predictions of M^*.

Adaptations of the Model

The Trend EOQ Model requires that the following be known and constant: the basic demand, the trend in demand, the ratio of cost parameters, and the length of the forecast period. The Trend EOQ Model also assumes that the cost parameters are separable from the demand parameter.[2]

One of the models presented in the discussion of the modifications of the Harris Model, in Chapter II, was found to be reducible to a constant multiplied by the square root of demand. This was the Non-Inventory EOQ Model. The model modified the Harris Model by consideration of the costs of a shortage. The Non-Inventory EOQ Model considered the balancing of shortage and ordering costs. If the shortage cost is known and constant and can be represented by π, in dollars

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[2]Assuming, as in Chapter III, that the demand in any decision period may be considered constant, application of the basic (Harris) model will yield:

$$q'_t = \sqrt{(2Sd_t)/(iU)} = \sqrt{(2S)/(iU)} \cdot \sqrt{d_t} = k \cdot \sqrt{d_t} \qquad (41)$$

The separation of cost parameters from the demand parameter refers to the capability of equation 45 to reduce to a constant, based upon the cost parameter ratio, multiplied by the square root of demand. This was implicit in the development of the Trend EOQ Model, and must be considered in any further development of adaptation of the model.

per unit per period, the economic order quantity for the Non-Inventory EOQ Model may be written as:

$$Q^* = \sqrt{\frac{2S}{\pi}} \cdot \sqrt{d} = \sqrt{(k\pi)_2} \cdot \sqrt{d} \tag{42}$$

Since the cost parameters are all considered to be known and constant, the cost ratio (S/π) will also be known and constant. The Trend EOQ Model will accommodate the Non-Inventory EOQ Model directly, since the cost parameter ratio will be a constant. It will only be necessary to substitute the correct value of the cost parameter ratio for (S/iU) in equation 30.

The Production EOQ Model assumes that receipt into stock does not occur instantaneously, but at some finite rate. If this finite rate is denoted by P and is a constant, the economic order quantity for the Production EOQ Model may be written as:

$$Q^* = \sqrt{\frac{2SdP}{iU(P - d)}} = \sqrt{\frac{2SP}{iU} \cdot \frac{d}{P - d}} = k_P \cdot \sqrt{\frac{d}{P - d}} \tag{43}$$

The economic order quantity for the Production EOQ Model may not be separated into a constant, based upon the cost parameter ratio, multiplied by the square root of demand, since the term $[(d)/(P - d)]$ may not be reduced further without re-introducing the demand parameter, d, into the cost ratio term. The Trend EOQ Model will not directly accomodate the Production EOQ Model under conditions of known demand-with-trend, since the $[(d)/(P - d)]$ term does not vary linearly with demand.

The Quantity Discount EOQ Models discussed in the references of Chapter II require that the constant economic order quantity be calculated and compared with the order quantity necessary to obtain the discount in price. If the economic order quantity is larger than the discount order quantity, the economic order quantity is ordered. If it is smaller, the costs of the policies based upon the EOQ and the discount order quantity are determined. The order quantity, economic or discount, with the lower-cost policy is ordered. This procedure must be followed for each discount or price-break.

Since the known demand-with-trend is assumed to be constant during any decision period, but varying throughout the forecast period, the Quantity Discount EOQ Model could not be combined with the Trend EOQ Model without necessitating one or more quantity and cost comparisons for each decision period. The number of comparisons would depend primarily upon the length of the forecast period and the number of price-breaks available, providing the economic order quantity for the first decision period did not exceed the largest discount order quantity. Assuming that this latter case did not occur, the study and implementation costs incurred through the use of the Trend EOQ Model would undoubtedly exceed the cost savings resulting from the application.

CHAPTER V

CONCLUSIONS

A change in the value of demand during the forecast period neccessitates a change in the economic order quantity, to be consistent with the criterion of cost minimization. The Trend EOQ Model presented in this study assumes that a constant change in demand occurs during the forecast period, but that the demand during any given decision period will be constant and be given by a linear demand function. The remainder of the assumptions of the Trend EOQ Model are those of the basic (Harris) model for determining the economic order quantity under the conditions of known and constant demand.

The Trend EOQ Model was developed from the concept that a linear order quantity function could be determined that would be consistent with the criterion of minimization of ordering and carrying costs and the criterion of maximization of the efficiency of the model application.

The Trend EOQ Model was compared against three other methods of determining the economic order quantities under the conditions of known demand-with-trend. The basis of comparison in each case was the forecast period total variable costs of the policies. The following conclusions may be drawn from the comparisons:

(1) The lowest-possible forecast period total variable costs will result from the use of the Harris Model to calculate the economic order quantity for each individual decision period. This policy will require one tedious square root calculation for each decision period

within the forecast period.

(2) The maximum forecast period total variable costs, among the policies considered, will result from the use of the Harris Model based upon the total demand for the forecast period. This is, in effect, the situation which would occur whenever the Harris Model is applied using the total forecast period demand under the false assumption that the demand was known and constant, rather than known-with-trend. The magnitude of the increase in forecast period total variable costs from the lowest-possible costs to the maximum costs will depend primarily upon the trend in demand and the length of the forecast period.

(3) Application of the concept that a linear order quantity function may be determined which will give the lowest forecast period total variable costs, among all such functions, will yield a forecast period total variable cost extremely near to the lowest-possible cost. The difference between costs is negligible, in most situations. This policy would require many calculations to determine the economic linear order quantity policy.

(4) The Trend EOQ Model provides a method for the prediction of the variable portion of the economic linear order quantity function. The change in the economic order quantity for each decision period, necessitated by the trend in demand, is predicted through the use of a linear-logarithmic multiple regression equation. The independent variables of this equation are: the basic demand, in units; the trend in demand, expressed as a percentage of the basic demand; the ratio of cost parameters; and the length of the forecast period, in decision periods. The linear-logarithmic multiple regression equation was found

to be sufficiently accurate to predict the optimum, integer-valued change in the economic order quantity with a minimum of deviation. This deviation from the optimum integer-valued change in the economic order quantity was not found to be sufficient to cause a significant increase in the forecast period total variable costs, over those obtained using the actual optimum change or over the lowest-possible forecast period total variable costs.

Three modifications of the basic (Harris) model were presented. Each of these three EOQ models retained the assumption that the forecast period demand was known and constant, but modified one of the remaining assumptions of the basic model. These three EOQ models were investigated to determine if the modifications were compatible with the assumptions and concepts of the Trend EOQ Model. The following further conclusions concerning the Trend EOQ Model were drawn during the course of the investigation:

(5) The Trend EOQ Model developed in the study will not directly accommodate the Production EOQ Model.

(6) The Trend EOQ Model will accommodate the rules presented for the Quantity Discount EOQ Model, but the number of additional calculations required may increase the study and implementation costs so that they will exceed the cost savings.

(7) The Trend EOQ Model will directly accommodate the Non-Inventory EOQ Model under known demand-with-trend. The method of calculating the cost ratio need only be changed to that method used under the basic Non-Inventory EOQ Model.

The following additional conclusions may be drawn from the entire study:

(8) Application of the Harris EOQ Model when the demand is actually non-constant will cause the total variable costs of the inventory policy to increase from their minimum. The resulting policy can not be considered "economic" or "optimum". The magnitude of this increase in policy costs will be greatest with a sharp increase in the demand for the item and with an extremely long forecast period.

(9) Further increases in total variable costs will be noted when groups of inventory items are considered, as in the Aggregate Inventory Policy presented by Welch (3). Items whose demands are not known and constant should not be included in such aggregate analyses. The greater the number of items having non-constant demand, the greater will be the increase in total variable costs over their optimum value.

CHAPTER VI

RECOMMENDATIONS FOR FURTHER STUDY

This study represents an initial investigation into the possibility of removing the most restrictive of the five major assumptions underlying the basic (Harris) EOQ Model. This assumption is: the forecast period demand must be known and constant. Since this is an initial investigation, many areas for further investigation have been disclosed. These areas are presented in the recommendations which follow:

(1) The mathematical formulation of the forecast period total variable cost curve for the integer-valued, linear order quantity function would bear further investigation. Perhaps an indirect or another approximate solution could be obtained. Such a solution should resemble the multiple regression equation developed in this study.

(2) Other forms of the equation for demand-with-trend should be considered. Under actual business conditions, the rate of change in demand would probably decrease with time and the decision period demand would probably approach an asymptotic limit. Forecasting techniques such as weighted averages or exponential smoothing should be investigated and the Trend EOQ Model expanded to include these techniques.

(3) The two remaining classes of demand, seasonal and seasonal-with-trend, should be considered and the Trend EOQ Model expanded to

include these classes.

(4) Further investigation should be directed toward the modification of the Trend EOQ Model to accommodate the assumption that receipt into stock occurs at some finite rate, rather than instantaneously. The results of any investigation of other forms of known demand-with-trend, with regard to the Trend EOQ Model, might prove useful in this context.

APPENDIX

Derivation: Equation 3, Page 13

$$(tvc) = \frac{d}{Q}(S) + \frac{Q}{2}(i)(U)$$

Taking the first derivative of (tvc) with respect to Q; setting it equal to zero; solving for Q; and calling the solution Q^*, the following result is obtained:

$$0 = -\frac{dS}{Q^2} + \frac{iU}{2} \ ,$$

$$\frac{iU}{2} = \frac{dS}{Q^2} \ ,$$

$$Q^2 = \frac{2dS}{iU} \ ,$$

and

$$Q = \sqrt{\frac{2dS}{iU}} = Q^*$$

To determine whether Q^* gives (tvc) a minimum or maximum value, take the second derivative of (tvc) with respect to Q. (tvc) will be a minimum if the second derivative is positive.

$$\text{Second Derivative} = +\frac{2dS}{Q^3} + 0 = \text{a positive number.}$$

Derivation: Equation 4, Page 13

$$(TVC) = \frac{D}{Q}(S) + \frac{Q}{2}(I)(U)$$

Taking the first derivative of (TVC) with respect to Q; setting it equal to zero; solving for Q; and calling the solution Q^*, the following result is obtained:

$$0 = -\frac{DS}{Q^2} + \frac{IU}{2}$$

$$\frac{IU}{2} = \frac{DS}{Q^2}$$

$$Q^2 = \frac{2DS}{IU}$$

and

$$Q = \sqrt{\frac{2DS}{IU}} = Q^*$$

To determine whether Q^* gives (TVC) a minimum or maximum value, take the second derivative of (TVC) with respect to Q. (TVC) will be a minimum if the second derivative is positive.

$$\text{Second Derivative} = +\frac{2DS}{Q^3} + 0 = \text{a positive number}$$

Derivation: Equation 14, Page 22

$$(tvc)' = \frac{(B + At)(S)}{(k \cdot \sqrt{B} + mt)} + \frac{(k \cdot \sqrt{B} + mt)(iU)}{2}$$

$$\frac{d(tvc)'}{dm} = -\frac{(B + At)(S)(t)}{(k \cdot \sqrt{B} + mt)^2} + \frac{(iU)(t)}{2}$$

Set $d(tvc)'/dm$ equal to zero; solve for m; and call this value m_t^*.

$$-\frac{(B + At)(S)(t)}{(k \cdot \sqrt{B} + mt)^2} + \frac{(iU)(t)}{2} = 0$$

$$(k \cdot \sqrt{B} + mt)^2 = (2S)(B + At)/(iU)$$

$$(k \cdot \sqrt{B} + mt) = \sqrt{(2S)/(iU)} \cdot \sqrt{B + At} = k \cdot \sqrt{B + At}$$

$$mt = k \cdot \sqrt{B + At} - k \cdot \sqrt{B}$$

$$m = k \cdot \frac{(\sqrt{B + At} - \sqrt{B})}{t} = m_t^*$$

Sample Calculations: Table 2, page 24.

Using the data from the example for the first decision period, $t = 1$, and equation 10, the optimal economic order quantity, $Q_1'^*$, is found to be:

$$Q_1'^* = k \cdot \sqrt{B + A(1)} = k \cdot \sqrt{500 + 50}$$

$$Q_1'^* = (22.36068)(23.45208) = 524.404 \text{ units}$$

The economic number of orders per decision period, $n_t'^*$, is found by dividing the decision period demand by $Q_t'^*$. For this example, $n_1'^*$ is found to be:

$$n_1'^* = (550)/(524.404) = 1.0488 \text{ orders}$$

The economic trend correction factor, m_t^*, is given by equation 14. For the example, m_1^* is found to be:

$$m_1^* = k \cdot \frac{(\sqrt{550} - \sqrt{500})}{1}$$

$$m_1^* = (22.36068)(23.45208 - 22.36068)$$

$$m_1^* = (22.36068)(1.09140) = 24.404 \text{ units.}$$

Calculation: $(TVC)'^*$ For: $d_t = 500 + 50t$

The optimum forecast period total variable cost shown on page 24 was calculated by use of a modified version of equation 2 and the totals of $Q_t'^*$ and $n_t'^*$ from Table 2. For the example, $(TVC)'^*$ was found to be:

$$(TVC)'^* = (S)\left(\sum_{1}^{12} n_t'^*\right) + (\tfrac{1}{2})\left(\sum_{1}^{12} Q_t'^*\right)(i)(U)$$

$$(TVC)'^* = (\$10.00)(15.3275) + (\tfrac{1}{2})(7{,}633.889)(0.02)(\$2.00)$$

$$(TVC)'^* = (\$153.28) + (\$153.28) = \$306.56$$

Calculations: Q_F' and $(TVC)_F'$ For: Example, page 28

Q_F' and $(TVC)_F'$ are calculated using equations 2 and 4, respectively, and the total forecast period demand given in Table 2. For the example, Q_F' and $(TVC)_F'$ are found to be:

$$Q_F' = \sqrt{\frac{(2)(\$10.00)(9{,}900)}{(12)(0.02)(\$2.00)}} = \sqrt{412{,}500} = 642.26 \text{ units}$$

and

$$(TVC)_F' = (\$10.00)(9{,}900/642.26) + (\tfrac{1}{2})(642.26)(0.02)(12)(\$2.00)$$

$$(TVC)_F' = (\$154.14) + (\$154.14) = \$308.28.$$

Derivation: Table 3, page 25

t	d_t	$Q'_M = 500 + 5t$		$Q'_M = 500 + 10t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	505	1.0891	510	1.0784
2	600	510	1.1765	520	1.1538
3	650	515	1.2621	530	1.2264
4	700	520	1.3462	540	1.2963
5	750	525	1.4286	550	1.3636
6	800	530	1.5094	560	1.4286
7	850	535	1.5888	570	1.4912
8	900	540	1.6667	580	1.5517
9	950	545	1.7431	590	1.6102
10	1,000	550	1.8182	600	1.6667
11	1,050	555	1.8919	610	1.7213
12	1,100	560	1.9643	620	1.7742
Totals	9,900	6,390	18.4849	6,780	17.3624

t	d_t	$Q'_M = 500 + 15t$		$Q'_M = 500 + 18t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	515	1.0680	518	1.0618
2	600	530	1.1321	536	1.1194
3	650	545	1.1927	554	1.1733
4	700	560	1.2500	572	1.2238
5	750	575	1.3043	590	1.2712
6	800	590	1.3559	608	1.3158
7	850	605	1.4050	626	1.3578
8	900	620	1.4516	644	1.3975
9	950	635	1.4961	662	1.4350
10	1,000	650	1.5385	680	1.4706
11	1,050	665	1.5789	698	1.5043
12	1,100	680	1.6176	716	1.5363
Totals	9,900	7,170	16.3907	7,404	15.8668

Derivation: Table 3, page 25 (continued)

t	d_t	$Q'_M = 500 + 19t$		$Q'_M = 500 + 20t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	519	1.0597	520	1.0577
2	600	538	1.1152	540	1.1111
3	650	557	1.1670	560	1.1607
4	700	576	1.2153	580	1.2069
5	750	595	1.2605	600	1.2500
6	800	614	1.3029	620	1.2903
7	850	633	1.3428	640	1.3281
8	900	652	1.3804	660	1.3636
9	950	671	1.4158	680	1.3971
10	1,000	690	1.4493	700	1.4286
11	1,050	709	1.4810	720	1.4583
12	1,100	728	1.5110	740	1.4865
Totals	9,900	7,482	15.7009	7,560	15.5389

t	d_t	$Q'_M = 500 + 21t$		$Q'_M = 500 + 22t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	521	1.0557	522	1.0536
2	600	542	1.1070	544	1.1029
3	650	563	1.1545	566	1.1484
4	700	584	1.1986	588	1.1905
5	750	605	1.2397	610	1.2295
6	800	626	1.2780	632	1.2658
7	850	647	1.3138	654	1.2997
8	900	668	1.3473	676	1.3314
9	950	689	1.3788	698	1.3610
10	1,000	710	1.4085	720	1.3889
11	1,050	731	1.4364	742	1.4151
12	1,100	752	1.4626	764	1.4399
Totals	9,900	7,638	15.3809	7,716	15.2267

Derivation: Table 3, page 25 (continued)

t	d_t	$Q'_M = 500 + 23t$		$Q'_M = 500 + 24t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	523	1.0516	524	1.0496
2	600	546	1.0989	548	1.0949
3	650	569	1.1424	572	1.1364
4	700	592	1.1824	596	1.1745
5	750	615	1.2195	620	1.2097
6	800	638	1.2539	644	1.2422
7	850	661	1.2859	668	1.2725
8	900	684	1.3158	692	1.3006
9	950	707	1.3437	716	1.3268
10	1,000	730	1.3699	740	1.3514
11	1,050	753	1.3944	764	1.3743
12	1,100	776	1.4176	788	1.3959
Totals	9,900	7,794	15.0760	7,872	14.9288

t	d_t	$Q'_M = 500 + 25t$		$Q'_M = 500 + 30t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	525	1.0476	530	1.0377
2	600	550	1.0909	560	1.0714
3	650	575	1.1304	590	1.1017
4	700	600	1.1667	620	1.1290
5	750	625	1.2000	650	1.1538
6	800	650	1.2308	680	1.1765
7	850	675	1.2593	710	1.1972
8	900	700	1.2857	740	1.2162
9	950	725	1.3103	770	1.2338
10	1,000	750	1.3333	800	1.2500
11	1,050	775	1.3548	830	1.2651
12	1,100	800	1.3751	860	1.2791
Totals	9,900	7,950	14.7849	8,340	14.1115

Derivation: Table 3, page 25 (continued)

t	d_t	$Q'_M = 500 + 35t$		$Q'_M = 500 + 40t$	
		Q'_M	n'_M	Q'_M	n'_M
1	550	535	1.0280	540	1.0185
2	600	570	1.0526	580	1.0345
3	650	605	1.0744	620	1.0484
4	700	640	1.0938	660	1.0606
5	750	675	1.1111	700	1.0714
6	800	710	1.1268	740	1.0811
7	850	745	1.1409	780	1.0897
8	900	780	1.1538	820	1.0976
9	950	815	1.1656	860	1.1047
10	1,000	850	1.1765	900	1.1111
11	1,050	885	1.1864	940	1.1170
12	1,100	920	1.1957	980	1.1224
Totals	9,900	8,730	13.5056	9,120	12.9570

Derivation: Table 3, page 25 (concluded)

$$(TVC)'_5 = (18.4849)(\$10.00) + \tfrac{1}{2}(6,390)(0.02)(\$2.00) = \$312.65$$

$$(TVC)'_{10} = (17.3624)(\$10.00) + \tfrac{1}{2}(6,780)(0.02)(\$2.00) = \$309.22$$

$$(TVC)'_{15} = (16.3907)(\$10.00) + \tfrac{1}{2}(7,170)(0.02)(\$2.00) = \$307.31$$

$$(TVC)'_{18} = (15.8668)(\$10.00) + \tfrac{1}{2}(7,404)(0.02)(\$2.00) = \$306.75$$

$$(TVC)'_{19} = (15.7009)(\$10.00) + \tfrac{1}{2}(7,482)(0.02)(\$2.00) = \$306.65$$

$$(TVC)'_{20} = (15.5389)(\$10.00) + \tfrac{1}{2}(7,560)(0.02)(\$2.00) = \$306.59$$

$$(TVC)'_{21} = (15.3809)(\$10.00) + \tfrac{1}{2}(7,638)(0.02)(\$2.00) = \$306.57$$

$$(TVC)'_{22} = (15.2267)(\$10.00) + \tfrac{1}{2}(7,716)(0.02)(\$2.00) = \$306.59$$

$$(TVC)'_{23} = (15.0760)(\$10.00) + \tfrac{1}{2}(7,794)(0.02)(\$2.00) = \$306.64$$

$$(TVC)'_{24} = (14.9288)(\$10.00) + \tfrac{1}{2}(7,872)(0.02)(\$2.00) = \$306.73$$

$$(TVC)'_{25} = (14.7849)(\$10.00) + \tfrac{1}{2}(7,950)(0.02)(\$2.00) = \$306.85$$

$$(TVC)'_{30} = (14.1115)(\$10.00) + \tfrac{1}{2}(8,340)(0.02)(\$2.00) = \$307.92$$

$$(TVC)'_{35} = (13.5056)(\$10.00) + \tfrac{1}{2}(8,730)(0.02)(\$2.00) = \$309.66$$

$$(TVC)'_{40} = (12.9570)(\$10.00) + \tfrac{1}{2}(9,120)(0.02)(\$2.00) = \$311.97$$

Investigation of a Direct Solution for M^*

If the length of the forecast period is twelve decision periods and the order quantity is given by Q'_M, the equation for the forecast period total variable cost may be written as:

$$(TVC)'_M = (S) \cdot \left(\sum_{t=1}^{12} \frac{d_t}{Q'_M} \right) + (\tfrac{1}{2}) \cdot \left(\sum_{t=1}^{12} Q'_M \right) \cdot (iU)$$

$$(TVC)'_M = (S) \cdot \sum_{t=1}^{12} \frac{B + At}{k \cdot \sqrt{B} + Mt} + (\tfrac{1}{2}) \cdot \sum_{t=1}^{12} (k \cdot \sqrt{B} + Mt) \cdot (iU)$$

Evaluate $(TVC)'_M$ for: $t = 1, 2, \ldots, 12.$

$$(TVC)'_M = \left[\frac{(S)(B + 1A)}{(k \cdot \sqrt{B} + 1M)} + \frac{(S) \cdot (B + 2A)}{(k \cdot \sqrt{B} + 2M)} + \cdots + \frac{(S) \cdot (B + 12A)}{(k \cdot \sqrt{B} + 12M)} \right]$$

$$+ \left[(\tfrac{1}{2}) \cdot (iU) \cdot (12k \cdot \sqrt{B}) + (\tfrac{1}{2}) \cdot (iU)(78M) \right]$$

Take the first derivative of $(TVC)'_M$ with respect to M and set it equal to zero.

$$\frac{d(TVC)'_M}{dM} = 0 = \left[0 + (39)(iU) \right] +$$

$$\left[- \frac{(S)(B + 1A)(1)}{(k \cdot \sqrt{B} + 1M)^2} - \frac{(S)(B + 2A)(2)}{(k \cdot \sqrt{B} + 2M)^2} - \cdots - \frac{(S)(B + 12A)(12)}{(k \cdot \sqrt{B} + 12M)^2} \right]$$

Investigation of a Direct Solution for M^* (Continued)

It should be noted that taking the second derivative of $(TVC)'_M$ with respect to M would cause the first bracketed term of the preceding equation to go to zero. The second term in brackets would experience a sign change. Thus, the second derivative would be positive and the value of M found by solving the preceding equation would give $(TVC)'_M$ its minimum value.

To solve for M in the preceding equation, it would first be necessary to obtain a common denominator for the second bracketed term. The common denominator would be a 24-th degree expression in M, k, and B. It does not seem practicable to obtain such an exact solution for M^*, since the $(TVC)'_M$ curve is relatively flat near M^* and a close approximation to M^* will not increase the value of $(TVC)'_M$ substantially.

Calculations: Values of M^*

Parameter Subscripts				(M^*+1)		(M^*)		(M^*-1)	
B	A	(S/iU)	T		$(TVC)'_{(M^*+1)}$		$(TVC)'_{(M^*)}$		$(TVC)'_{(M^*-1)}$
1	1	1	1	12	$275.681	11	$275.672	10	$275.709
1	1	1	2	12	608.635	11	608.373	10	608.387
1	1	1	3	11	990.031	10	989.580	9	989.961
1	1	2	1	11	246.584	10	246.568	9	246.600
1	1	2	2	10	544.170	9	544.153	8	544.464
1	1	2	3	10	885.653	9	885.112	8	885.503
1	1	3	1	10	213.572	9	213.534	8	213.552
1	1	3	2	9	471.341	8	471.229	7	471.490
1	1	3	3	9	767.377	8	766.571	7	766.812
1	2	1	1	22	$306.586	21	$306.571	20	$306.589
1	2	1	2	20	711.285	19	711.140	18	711.230
1	2	1	3	18	1,194.281	17	1,194.207	16	1,194.827
1	2	2	1	20	274.232	19	274.204	18	274.219
1	2	2	2	18	636.218	17	636.066	16	636.171
1	2	2	3	17	1,068.831	16	1,068.162	15	1,068.213
1	2	3	1	17	237.479	16	237.471	15	237.512
1	2	3	2	16	551.116	15	550.868	14	550.912
1	2	3	3	14	925.111	13	925.100	12	925.996
1	3	1	1	31	$334.336	30	$334.310	29	$334.315
1	3	1	2	27	799.985	26	799.893	25	800.005
1	3	1	3	25	1,367.508	24	1,367.046	23	1,367.145
1	3	2	1	28	299.051	27	299.018	26	299.021
1	3	2	2	24	715.502	23	715.460	22	715.648
1	3	2	3	22	1,222.914	21	1,222.714	20	1,223.160
1	3	3	1	24	258.974	23	258.954	22	258.978
1	3	3	2	21	619.682	20	619.603	19	619.790
1	3	3	3	19	1,059.051	18	1,058.941	17	1,059.590
2	1	1	1	29	$675.267	28	$675.253	27	$675.257
2	1	1	2	27	1,490.214	26	1,490.136	25	1,490.172
2	1	1	3	25	2,424.033	24	2,423.979	23	2,424.273
2	1	2	1	26	603.977	25	603.964	24	603.972
2	1	2	2	24	1,332.867	23	1,332.820	22	1,332.902
2	1	2	3	23	2,168.333	22	2,168.069	21	2,168.186
2	1	3	1	22	523.051	21	523.050	20	523.072
2	1	3	2	21	1,154.328	20	1,154.252	19	1,154.326
2	1	3	3	20	1,877.878	19	1,877.594	18	1,877.752
2	2	1	1	52	$750.9399	51	$750.9397	50	$750.955
2	2	1	2	47	1,741.955	46	1,741.929	45	1,741.998
2	2	1	3	44	2,925.341	43	2,925.096	42	2,925.119
2	2	2	1	47	671.667	46	671.661	45	671.672
2	2	2	2	42	1,558.052	41	1,558.041	40	1,558.136
2	2	2	3	39	2,616.396	38	2,616.281	37	2,616.472

Calculations: Values of M^* (continued)

Parameter Subscripts B	A	(S/iU)	T	(M^*+1)	$(TVC)'_{(M^*+1)}$	(M^*)	$(TVC)'_{(M^*)}$	(M^*-1)	$(TVC)'_{(M^*-1)}$
2	2	3	1	41	$581.686	40	$581.674	39	$581.681
2	2	3	2	37	1,349.377	36	1,349.291	35	1,349.329
2	2	3	3	34	2,265.929	33	2,265.749	32	2,265.920
2	3	1	1	74	$818.893	73	$818.882	72	$818.885
2	3	1	2	65	1,959.387	64	1,959.329	63	1,959.352
2	3	1	3	59	3,348.610	58	3,348.485	57	3,348.593
2	3	2	1	66	732.438	65	732.432	64	732.442
2	3	2	2	58	1,752.522	57	1,752.486	56	1,752.540
2	3	2	3	53	2,995.162	52	2,995.001	51	2,995.098
2	3	3	1	57	634.307	56	634.303	55	634.318
2	3	3	2	50	1,517.705	49	1,517.694	48	1,517.791
2	3	3	3	46	2,593.904	45	2,593.723	44	2,593.842
3	1	1	1	39	$914.309	38	$914.297	37	$914.299
3	1	1	2	36	2,017.697	35	2,017.653	34	2,017.699
3	1	1	3	34	3,282.192	33	3,282.062	32	3,282.187
3	1	2	1	35	817.785	34	817.773	33	817.775
3	1	2	2	32	1,804.667	31	1,804.648	30	1,804.724
3	1	2	3	31	2,935.879	30	2,935.600	29	2,935.601
3	1	3	1	30	708.218	29	708.211	28	708.220
3	1	3	2	28	1,562.913	27	1,562.865	26	1,562.927
3	1	3	3	27	2,542.620	26	2,542.205	25	2,542.312
3	2	1	1	71	$1,016.784	70	$1,016.776	69	$1,016.780
3	2	1	2	64	2,358.650	63	2,358.590	62	2,358.599
3	2	1	3	59	3,960.752	58	3,960.590	57	3,960.629
3	2	2	1	63	909.434	62	909.433	61	909.444
3	2	2	2	57	2,109.620	56	2,109.582	55	2,109.623
3	2	2	3	53	3,542.669	52	3,542.469	51	3,542.490
3	2	3	1	55	787.598	54	787.590	53	787.597
3	2	3	2	49	1,826.958	48	1,826.955	47	1,827.044
3	2	3	3	46	3,068.069	45	3,067.853	44	3,067.893
3	3	1	1	100	$1,108.784	99	$1,108.7740	98	$1,108.7743
3	3	1	2	87	2,652.961	86	2,652.956	85	2,653.011
3	3	1	3	80	4,534.081	79	4,533.913	78	4,533.915
3	3	2	1	89	991.721	88	991.717	87	991.724
3	3	2	2	78	2,372.888	77	2,372.873	76	2,372.926
3	3	2	3	71	4,055.293	70	4,055.239	69	4,055.379
3	3	3	1	77	858.853	76	858.850	75	858.859
3	3	3	2	68	2,055.005	67	2,054.957	66	2,054.987
3	3	3	3	62	3,512.077	61	3,511.919	60	3,511.982

Calculations: Predicted Value of M^* and (DEV)

Parameter Subscripts				$\log(M_P^*)$	Unrounded M_P^*	M_P^*	M^*	(DEV)
B	A	(S/iU)	T					
1	1	1	1	1.06922	11.728	12	11	+1
1	1	1	2	1.02055	10.485	10	11	-1
1	1	1	3	0.99207	9.819	10	10	0
1	1	2	1	1.02013	10.474	10	10	0
1	1	2	2	0.97145	9.364	9	9	0
1	1	2	3	0.94298	8.770	9	9	0
1	1	3	1	0.95684	9.054	9	9	0
1	1	3	2	0.90816	8.094	8	8	0
1	1	3	3	0.87969	7.580	8	8	0
1	2	1	1	1.31891	20.840	21	21	0
1	2	1	2	1.27022	18.630	19	19	0
1	2	1	3	1.24175	17.448	17	17	0
1	2	2	1	1.26981	18.613	19	19	0
1	2	2	2	1.22113	16.639	17	17	0
1	2	2	3	1.19266	15.583	16	16	0
1	2	3	1	1.20652	16.089	16	16	0
1	2	3	2	1.15784	14.383	14	15	-1
1	2	3	3	1.12937	13.470	13	13	0
1	3	1	1	1.46495	29.171	29	30	-1
1	3	1	2	1.41627	26.078	26	26	0
1	3	1	3	1.38780	24.423	24	24	0
1	3	2	1	1.41586	26.053	26	27	-1
1	3	2	2	1.36718	23.291	23	23	0
1	3	2	3	1.33871	21.813	22	21	+1
1	3	3	1	1.35257	22.520	23	23	0
1	3	3	2	1.30389	20.132	20	20	0
1	3	3	3	1.27542	18.855	19	18	+1
2	1	1	1	1.45873	28.756	29	28	+1
2	1	1	2	1.41005	25.707	26	26	0
2	1	1	3	1.38158	24.076	24	24	0
2	1	2	1	1.40963	25.682	26	25	+1
2	1	2	2	1.36096	22.959	23	23	0
2	1	2	3	1.33248	21.502	22	22	0
2	1	3	1	1.34634	22.199	22	21	+1
2	1	3	2	1.29767	19.846	20	20	0
2	1	3	3	1.26919	18.586	19	19	0
2	2	1	1	1.70840	51.098	51	51	0
2	2	1	2	1.65973	45.680	46	46	0
2	2	1	3	1.63125	42.781	43	43	0
2	2	2	1	1.65931	45.636	46	46	0
2	2	2	2	1.61063	40.797	41	41	0

Calculations: Predicted Value of M^* and (DEV) (continued)

Parameter Subscripts B	A	(S/iU)	T	$\log(M^*_P)$	Unrounded M^*_P	M^*_P	M^*	(DEV)
2	2	2	3	1.58216	38.208	38	38	0
2	2	3	1	1.59602	39.447	39	40	-1
2	2	3	2	1.54734	35.265	35	36	-1
2	2	3	3	1.51887	33.027	33	33	0
2	3	1	1	1.85445	71.523	72	73	-1
2	3	1	2	1.80578	63.942	64	64	0
2	3	1	3	1.77730	59.883	60	58	+2
2	3	2	1	1.80536	63.880	64	65	-1
2	3	2	2	1.75669	57.107	57	57	0
2	3	2	3	1.72821	53.483	53	52	+1
2	3	3	1	1.74207	55.216	55	56	-1
2	3	3	2	1.69339	49.361	49	49	0
2	3	3	3	1.66492	46.230	46	45	+1
3	1	1	1	1.59049	38.948	39	38	+1
3	1	1	2	1.54181	34.818	35	35	0
3	1	1	3	1.51334	32.609	33	33	0
3	1	2	1	1.54140	34.786	35	34	+1
3	1	2	2	1.49272	31.097	31	31	0
3	1	2	3	1.46425	29.124	29	30	-1
3	1	3	1	1.47811	30.069	30	29	+1
3	1	3	2	1.42943	26.880	27	27	0
3	1	3	3	1.40096	25.174	25	26	-1
3	2	1	1	1.84017	69.210	69	70	-1
3	2	1	2	1.79149	61.871	62	63	-1
3	2	1	3	1.76302	57.946	58	58	0
3	2	2	1	1.79108	61.813	62	62	0
3	2	2	2	1.74240	55.259	55	56	-1
3	2	2	3	1.71393	51.753	52	52	0
3	2	3	1	1.72778	53.429	53	54	-1
3	2	3	2	1.67911	47.765	48	48	0
3	2	3	3	1.65063	44.733	45	45	0
3	3	1	1	1.98622	96.878	97	99	-2
3	3	1	2	1.93754	86.604	87	86	+1
3	3	1	3	1.90907	81.110	81	79	+2
3	3	2	1	1.93713	86.522	87	88	-1
3	3	2	2	1.88845	77.348	77	77	0
3	3	2	3	1.85998	72.440	72	70	+2
3	3	3	1	1.87384	74.790	75	76	-1
3	3	3	2	1.82516	66.859	67	67	0
3	3	3	3	1.79668	62.616	63	61	+2

Note: $\text{Log}(M^*_P)$ calculated using equation 30 and (DEV) was calculated using equation 31.

Sample Calculations: Table 5, page 37

The following calculations are based upon the $B_1A_2(S/iU)_1T_2$ observation used in the example. Equation 34 is used to determine the value of $\log(M_P^*)$.

$$\log(M_P^*) = -1.90172 + 0.50055\left[\log(B)\right] + 0.82941\left[\log(A)\right] + 0.50658\left[\log(S/iU)\right] - 0.16170\left[\log(T)\right]$$

$$\log(M_P^*) = -(1.90172) + (0.50055)(2.69897) + (0.82941)(1.00000) + (0.50658)(2.39794) - (0.16170)(1.07918)$$

$$\log(M_P^*) = -(1.90172) + (1.35097) + (0.82941) + (1.21475) - (0.17450)$$

$$\log(M_P^*) = 1.31891$$

$$M_P^* = (20.840)_{\text{unrounded}} = 21$$

Policy Comparison: Forecast Period Total Variable Costs

Parameter Subscripts							
B	A	(S/iU)	T	$(TVC)'_F$	$(TVC)'^*$	$(TVC)'_{M^*}$	$(TVC)'_{M^*_P}$
1	1	1	1	$276.26	$275.67	$275.67	$275.68
1	1	1	2	611.88	608.32	608.37	608.38
1	1	1	3	998.96	989.46	989.58	989.58
1	1	2	1	247.10	246.57	246.57	246.57
1	1	2	2	547.28	544.10	544.15	544.15
1	1	2	3	893.50	885.00	885.11	885.11
1	1	3	1	213.99	213.53	213.53	213.53
1	1	3	2	473.96	471.20	471.23	471.23
1	1	3	3	773.79	766.43	766.57	766.57
1	2	1	1	$308.28	$306.56	$306.57	$306.57
1	2	1	2	720.00	710.97	711.14	711.14
1	2	1	3	1,215.50	1,193.52	1,194.21	1,194.21
1	2	2	1	275.74	274.19	274.20	274.20
1	2	2	2	643.98	635.91	636.07	636.07
1	2	2	3	1,087.18	1,067.52	1,068.16	1,068.16
1	2	3	1	238.80	237.46	237.47	237.47
1	2	3	2	557.71	550.72	550.87	550.91
1	2	3	3	941.52	924.50	925.10	925.10
1	3	1	1	$337.28	$334.26	$334.31	$334.32
1	3	1	2	813.88	799.46	799.89	799.89
1	3	1	3	1,398.92	1,365.52	1,367.05	1,367.05
1	3	2	1	301.68	298.98	299.02	299.02
1	3	2	2	727.96	715.06	715.46	715.46
1	3	2	3	1,251.23	1,221.36	1,222.71	1,222.91
1	3	3	1	261.26	258.92	258.95	258.95
1	3	3	2	630.43	619.26	619.60	619.60
1	3	3	3	1,083.59	1,057.73	1,058.94	1,059.05
2	1	1	1	$676.69	$675.25	$675.25	$675.27
2	1	1	2	1,498.80	1,490.07	1,490.14	1,490.14
2	1	1	3	2,446.94	2,423.68	2,423.98	2,423.98
2	1	2	1	605.26	603.96	603.96	603.98
2	1	2	2	1,340.57	1,332.76	1,332.82	1,332.82
2	1	2	3	2,188.61	2.167.80	2,168.07	2,168.07
2	1	3	1	524.17	523.05	523.05	523.05
2	1	3	2	1,160.96	1,154.20	1,154.25	1,154.25
2	1	3	3	1,895.40	1,877.37	1,877.59	1,877.59
2	2	1	1	$755.14	$750.90	$750.94	$750.94
2	2	1	2	1,763.63	1,741.52	1,741.93	1,741.93
2	2	1	3	2,977.35	2,923.52	2,925.10	2,925.10
2	2	2	1	675.42	671.63	671.66	671.66
2	2	2	2	1,577.44	1,557.66	1,558.04	1,558.04
2	2	2	3	2,663.03	2,614.88	2,616.28	2,616.28

Policy Comparison: Forecast Period Total Variable Costs (continued)

Parameter Subscripts B	A	(S/iU)	T	$(TVC)'_F$	$(TVC)'^*$	$(TVC)'_{M^*}$	$(TVC)'_{M^*_P}$
2	2	3	1	$584.93	$581.65	$581.67	$581.68
2	2	3	2	1,366.10	1,348.98	1,349.29	1,349.33
2	2	3	3	2,306.25	2,264.55	2,265.75	2,265.75
2	3	1	1	$826.17	$818.78	$818.88	$818.89
2	3	1	2	1,993.59	1,958.37	1,959.33	1,959.33
2	3	1	3	3,426.62	3,344.83	3,348.49	3,348.96
2	3	2	1	738.95	732.34	732.43	732.44
2	3	2	2	1,783.12	1,751.53	1,752.49	1,752.49
2	3	2	3	3,064.87	2,991.71	2,995.00	2,995.16
2	3	3	1	639.95	634.22	634.30	634.32
2	3	3	2	1,544.23	1,516.87	1,517.69	1,517.69
2	3	3	3	2,654.25	2,590.89	2,593.72	2,593.90
3	1	1	1	$916.25	$914.29	$914.30	$914.31
3	1	1	2	2,029.38	2,017.57	2,017.65	2,017.65
3	1	1	3	3,313.18	3,281.67	3,282.06	3,282.06
3	1	2	1	819.52	817.77	817.77	817.79
3	1	2	2	1,815.13	1,804.57	1,804.65	1,804.65
3	1	2	3	2,963.39	2,935.22	2,935.60	2,935.60
3	1	3	1	709.72	708.21	708.21	708.22
3	1	3	2	1,571.95	1,562.80	1,562.87	1,562.87
3	1	3	3	2,566.37	2,541.97	2,542.21	2,542.31
3	2	1	1	$1,022.47	$1,016.73	$1,016.78	$1,016.78
3	2	1	2	2,387.97	2,358.03	2,358.59	2,358.60
3	2	1	3	4,031.36	3,958.47	3,960.59	3,960.59
3	2	2	1	914.52	909.39	909.43	909.43
3	2	2	2	2,135.86	2,109.09	2,109.58	2,109.62
3	2	2	3	3,605.76	3,540.56	3,542.47	3,542.47
3	2	3	1	792.00	787.56	787.59	787.60
3	2	3	2	1,849.71	1,826.52	1,826.96	1,826.96
3	2	3	3	3,122.68	3,066.22	3,067.85	3,067.85
3	3	1	1	$1,118.64	$1,108.63	$1,108.77	$1,109.19
3	3	1	2	2,699.33	2,651.51	2,652.96	2,652.96
3	3	1	3	4,639.67	4,528.92	4,533.91	4,534.42
3	3	2	1	1,000.54	991.59	991.72	991.72
3	3	2	2	2,414.36	2,371.58	2,372.87	2,372.87
3	3	2	3	4,149.85	4,050.79	4,055.24	4,055.54
3	3	3	1	866.50	858.74	858.85	858.86
3	3	3	2	2,090.89	2,053.85	2,054.96	2,054.96
3	3	3	3	3,593.88	3,508.09	3,511.92	3,512.45

Policy Comparisons: Maximum, Minimum, and Expected Losses

Parameter Subscripts				Maximum Loss		Minimum Loss		Expected Loss	
B	A	(S/iU)	T	Dollars	Per Cent	Dollars	Per Cent	Dollars	Per Cent
1	1	1	1	$0.59	0.21%	$0.00	0.00%	$0.01	0.00%
1	1	1	2	3.56	0.59	0.05	0.01	0.06	0.01
1	1	1	3	9.50	0.96	0.12	0.01	0.12	0.01
1	1	2	1	0.53	0.21	0.00	0.00	0.00	0.00
1	1	2	2	3.18	0.58	0.05	0.01	0.05	0.01
1	1	2	3	8.50	0.96	0.11	0.01	0.11	0.01
1	1	3	1	0.46	0.22	0.00	0.00	0.00	0.00
1	1	3	2	2.76	0.59	0.03	0.01	0.03	0.01
1	1	3	3	7.36	0.96	0.14	0.02	0.14	0.02
1	2	1	1	$1.72	0.56%	$0.01	0.00%	$0.01	0.00%
1	2	1	2	9.03	1.27	0.17	0.02	0.17	0.02
1	2	1	3	21.98	1.84	0.69	0.06	0.69	0.06
1	2	2	1	1.55	0.57	0.01	0.00	0.01	0.00
1	2	2	2	8.07	1.27	0.16	0.03	0.16	0.03
1	2	2	3	19.66	1.84	0.64	0.06	0.64	0.06
1	2	3	1	1.34	0.56	0.01	0.00	0.01	0.00
1	2	3	2	6.99	1.27	0.15	0.03	0.19	0.03
1	2	3	3	17.02	1.84	0.60	0.06	0.60	0.06
1	3	1	1	$3.02	0.90%	$0.05	0.01%	$0.06	0.02%
1	3	1	2	14.42	1.80	0.43	0.05	0.43	0.05
1	3	1	3	33.40	2.45	1.53	0.11	1.53	0.11
1	3	2	1	2.70	0.90	0.04	0.01	0.04	0.01
1	3	2	2	12.90	1.80	0.40	0.06	0.40	0.06
1	3	2	3	29.87	2.45	1.35	0.11	1.55	0.13
1	3	3	1	2.34	0.90	0.03	0.01	0.03	0.01
1	3	3	2	11.17	1.80	0.34	0.05	0.34	0.05
1	3	3	3	25.86	2.44	1.21	0.11	1.32	0.12
2	1	1	1	$1.44	0.21%	$0.00	0.00%	$0.02	0.00%
2	1	1	2	8.73	0.59	0.07	0.00	0.07	0.00
2	1	1	3	23.26	0.96	0.30	0.01	0.30	0.01
2	1	2	1	1.30	0.22	0.00	0.00	0.02	0.00
2	1	2	2	7.81	0.59	0.06	0.00	0.06	0.00
2	1	2	3	20.81	0.96	0.27	0.01	0.27	0.01
2	1	3	1	1.12	0.21	0.00	0.00	0.00	0.00
2	1	3	2	6.76	0.59	0.05	0.00	0.05	0.00
2	1	3	3	18.03	0.96	0.22	0.01	0.22	0.01
2	2	1	1	$4.24	0.56%	$0.04	0.01%	$0.04	0.01%
2	2	1	2	22.11	1.27	0.41	0.02	0.41	0.02
2	2	1	3	53.83	1.84	1.58	0.05	1.58	0.05
2	2	2	1	3.79	0.56	0.03	0.00	0.03	0.00
2	2	2	2	19.78	1.27	0.38	0.02	0.38	0.02
2	2	2	3	48.15	1.84	1.40	0.05	1.40	0.05

Policy Comparisons: Maximum, Minimum, and Expected Losses (continued)

Parameter Subscripts				Maximum Loss		Minimum Loss		Expected Loss	
B	A	(S/iU)	T	Dollars	Per Cent	Dollars	Per Cent	Dollars	Per Cent
2	2	3	1	$3.28	0.56%	$0.02	0.00%	$0.03	0.01%
2	2	3	2	17.12	1.27	0.31	0.02	0.35	0.03
2	2	3	3	41.70	1.84	1.20	0.05	1.20	0.05
2	3	1	1	7.39	0.90	0.10	0.01	0.11	0.01
2	3	1	2	35.22	1.80	0.96	0.05	0.96	0.05
2	3	1	3	81.79	2.45	3.66	0.11	4.13	0.12
2	3	2	1	6.61	0.90	0.09	0.01	0.10	0.01
2	3	2	2	31.59	1.80	0.96	0.05	0.96	0.05
2	3	2	3	73.16	2.45	3.29	0.11	3.45	0.12
2	3	3	1	5.73	0.90	0.08	0.01	0.10	0.02
2	3	3	2	27.36	1.80	0.82	0.05	0.82	0.05
2	3	3	3	63.36	2.45	2.83	0.11	3.01	0.12
3	1	1	1	$1.96	0.21%	$0.01	0.00%	$0.02	0.00%
3	1	1	2	11.81	0.59	0.08	0.00	0.08	0.00
3	1	1	3	31.51	0.96	0.39	0.01	0.39	0.01
3	1	2	1	1.75	0.21	0.00	0.00	0.02	0.00
3	1	2	2	10.56	0.59	0.08	0.00	0.08	0.00
3	1	2	3	28.17	0.96	0.38	0.01	0.38	0.01
3	1	3	1	1.51	0.21	0.00	0.00	0.01	0.00
3	1	3	2	9.15	0.59	0.07	0.00	0.07	0.00
3	1	3	3	24.40	0.96	0.24	0.01	0.34	0.01
3	2	1	1	$5.74	0.56%	$0.05	0.00%	$0.05	0.00%
3	2	1	2	29.94	1.27	0.56	0.02	0.57	0.02
3	2	1	3	72.89	1.84	2.12	0.05	2.12	0.05
3	2	2	1	5.13	0.56	0.04	0.00	0.04	0.00
3	2	2	2	26.77	1.27	0.49	0.02	0.53	0.03
3	2	2	3	65.20	1.84	1.91	0.05	1.91	0.05
3	2	3	1	4.44	0.56	0.03	0.00	0.04	0.01
3	2	3	2	23.19	1.27	0.44	0.02	0.44	0.02
3	2	3	3	56.46	1.84	1.63	0.05	1.63	0.05
3	3	1	1	$10.01	0.90%	$0.14	0.01%	$0.56	0.05%
3	3	1	2	47.82	1.80	1.45	0.05	1.45	0.05
3	3	1	3	110.75	2.45	4.99	0.11	5.50	0.12
3	3	2	1	8.95	0.90	0.13	0.01	0.13	0.01
3	3	2	2	42.78	1.80	1.29	0.05	1.29	0.05
3	3	2	3	99.06	2.45	4.45	0.11	4.75	0.12
3	3	3	1	7.76	0.90	0.11	0.01	0.12	0.01
3	3	3	2	37.04	1.80	1.11	0.05	1.11	0.05
3	3	3	3	85.79	2.45	3.83	0.11	4.36	0.12

Maximum Loss: $(TVC)'_F - (TVC)'^*$. Expected Loss: $(TVC)'_{M_P^*} - (TVC)'^*$.

Minimum Loss: $(TVC)'_{M^*} - (TVC)'^*$. Base for percentages: $(TVC)'^* = 100\%$

Model Effectiveness

Equation 37 of the text may also be presented in terms of forecast period total variable costs, as follows:

Model Effectiveness =

$$\left[\frac{(\text{Maximum Loss}) - (\text{Expected Loss})}{(\text{Maximum Loss})}\right] \cdot (100\%)$$

Model Effectiveness =

$$\left[\frac{\left[(TVC)'_F - (TVC)'^*\right] - \left[(TVC)'_{M_P^*} - (TVC)'^*\right]}{(TVC)'_F - (TVC)'^*}\right] \cdot (100\%)$$

Model Effectiveness =

$$\left[\frac{(TVC)'_F - (TVC)'_{M_P^*}}{(TVC)'_F - (TVC)'^*}\right] \cdot (100\%) .$$

Model Test #2

$B = 3{,}500$ units per decision period

$A =$ 6% of B per decision period (210 units per decision period)

$(S/iU) = [(\$9.60)/(0.02)(\$2.00)] = 240$

$T = 15$ decision periods per forecast period

$$\log(M_P^*) = -1.90172 + (0.50055)(3.54407) + (0.82941)(0.77815)$$
$$+ (0.50658)(2.38021) - (0.16170)(1.17609)$$

$\log(M_P^*) = 1.53327$

$M_P^* = (34.141)_{\text{unrounded}} = (34)_{\text{rounded}} = 34$

$(TVC)'_{M_P^*} = \$942.43$

$(TVC)'_{M^*+1} = (TVC)'_{35} = \$942.44$

$(TVC)'_{M^*} = (TVC)'_{34} = \$942.43$

$(TVC)'_{M^*-1} = (TVC)'_{33} = \$942.44$

$M^* = 34$

$(DEV) = 34 - 34 = 0$

$(TVC)'^* = \$942.41$

$(TVC)'_F = \$946.10$

Maximum Loss = \$946.10 - \$942.41 = \$3.69 = 0.39% of $(TVC)'^*$

Minimum Loss = \$942.43 - \$942.41 = \$0.02 = 0.00% of $(TVC)'^*$

Expected Loss = \$942.43 - \$942.41 = \$0.02 = 0.00% of $(TVC)'^*$

Model Effectiveness = $[(\$3.69 - \$0.02)/(\$3.69)] \cdot (100\%) = 99\%$

Model Test #3

B = 4,300 units per decision period

A = 7% of B per decision period (301 units per decision period)

(S/iU) = [($9.00)/(0.005)($8.00)] = 225

T = 17 decision periods per forecast period

$$\log(M_P^*) = -1.90172 + (0.50055)(3.63347) + (0.82941)(0.84510) + (0.50658)(2.35218) - (0.16170)(1.23045)$$

$$\log(M_P^*) = 1.61055$$

$$M_P^* = (40.790)_{\text{unrounded}} = (41)_{\text{rounded}} = 41$$

$$(TVC)'_{M_P^*} = \$1,200.85$$

$$(TVC)'_{M^*+1} = (TVC)'_{42} = \$1,200.87$$

$$(TVC)'_{M^*} = (TVC)'_{41} = \$1,200.85$$

$$(TVC)'_{M^*-1} = (TVC)'_{40} = \$1,200.87$$

$$M^* = 41$$

$$(DEV) = 41 - 41 = 0$$

$$(TVC)'^* = \$1,200.80$$

$$(TVC)'_F = \$1,207.66$$

Maximum Loss = \$1,207.66 - \$1,200.80 = \$6.86 = 0.57% of $(TVC)'^*$

Minimum Loss = \$1,200.85 - \$1,200.80 = \$0.05 = 0.00% of $(TVC)'^*$

Expected Loss = \$1,200.85 - \$1,200.80 = \$0.05 = 0.00% of $(TVC)'^*$

Model Effectiveness = [(\$6.86 - \$0.05)/(\$6.86)] · (100%) = 99%

Model Test #4

B = 2,500 units per decision period

A = 24% of B per decision period (600 units per decision period)

(S/iU) = [($17.00/(0.02($5.00)] = 170

T = 6 decision periods per forecast period

$$\log(M_P^*) = -1.90172 + (0.50055)(3.39794) + (0.82941)(1.38021)$$
$$+ (0.50658)(2.23045) - (0.16170)(0.77815)$$

$\log(M_P^*) = 1.94795$

$M_P^* = (88.706)_{\text{unrounded}} = (89)_{\text{rounded}} = 89$

$(TVC)'_{M_P^*} = \$745.64$

$(TVC)'_{M^*+1} = (TVC)'_{91} = \$745.64$

$(TVC)'_{M^*} = (TVC)'_{90} = \$745.63$

$(TVC)'_{M^*-1} = (TVC)'_{89} = \$745.64$

$M^* = 90$

$(DEV) = 89 - 90 = -1$

$(TVC)'^* = \$745.57$

$(TVC)'_F = \$750.36$

Maximum Loss = \$750.36 - \$745.57 = \$4.79 = 0.64% of $(TVC)'^*$

Minimum Loss = \$745.63 - \$745.57 = \$0.06 = 0.01% of $(TVC)'^*$

Expected Loss = \$745.64 - \$745.57 = \$0.07 = 0.01% of $(TVC)'^*$

Model Effectiveness = [(\$4.79 - \$0.07)/(\$4.79)] · (100%) = 99%

Model Test #5

$B = 6{,}000$ units per decision period

$A = 2.5\%$ of B per decision period (150 units per decision period)

$(S/iU) = \left[(\$28.00)/(0.005)(\$35.00)\right] = 160$

$T = 50$ decision periods per forecast period

$$\log(M^*_P) = -1.90172 + (0.50055)(3.77815) + (0.82941)(0.39794)$$
$$+ (0.50658)(2.20412) - (0.16170)(1.69897)$$

$$\log(M^*_P) = 1.16133$$

$$M^*_P = (14.499)_{\text{unrounded}} = (14)_{\text{rounded}} = 14$$

$$(TVC)'_{M^*_P} = \$15{,}419.31$$

$$(TVC)'_{M^*+1} = (TVC)'_{16} = \$15{,}422.78$$

$$(TVC)'_{M^*} = (TVC)'_{15} = \$15{,}419.15$$

$$(TVC)'_{M^*-1} = (TVC)'_{14} = \$15{,}419.31$$

$$M^* = 15$$

$$(DEV) = 14 - 15 = -1$$

$$(TVC)'^* = \$15{,}418.06$$

$$(TVC)'_F = \$15{,}514.91$$

Maximum Loss = $\$15{,}514.91 - \$15{,}418.06 = \$96.85 = 0.63\%$ of $(TVC)'^*$

Minimum Loss = $\$15{,}419.15 - \$15{,}418.06 = \$1.09 = 0.01\%$ of $(TVC)'^*$

Expected Loss = $\$15{,}419.31 - \$15{,}418.06 = \$1.25 = 0.01\%$ of $(TVC)'^*$

Model Effectiveness = $\left[(\$96.85 - \$1.25)/(\$96.85)\right] \cdot (100\%) = 99\%$

Model Test #6

B = 8,000 units per decision period

A = 24% of B per decision period (1,920 units per decision period)

$(S/iu) = \left[(\$24.00)/(0.02)(\$4.00)\right] = 300$

T = 6 decision periods per forecast period

$$\log(M_p^*) = -1.90172 + (0.50055)(3.90309) + (0.82941)(1.38021)$$
$$+ (0.50658)(2.47712) - (0.16170)(0.77815)$$

$$\log(M_p^*) = 2.32576$$

$$M_p^* = (211.72)_{\text{unrounded}} = (212)_{\text{rounded}} = 212$$

$$(TVC)'_{M_p^*} = \$1,417.52$$

$$(TVC)'_{M^*+1} = (TVC)'_{216} = \$1,417.515$$

$$(TVC)'_{M^*} = (TVC)'_{215} = \$1,417.512$$

$$(TVC)'_{M^*-1} = (TVC)'_{214} = \$1,417.513$$

$$M^* = 215$$

$$(DEV) = 212 - 215 = -3$$

$$(TVC)'^* = \$1,417.40$$

$$(TVC)'_F = \$1,426.50$$

Maximum Loss = \$1,426.50 - \$1,417.40 = \$9.10 = 0.64% of $(TVC)'^*$

Minimum Loss = \$1,417.51 - \$1,417.40 = \$0.11 = 0.01% of $(TVC)'^*$

Expected Loss = \$1,417.52 - \$1,417.40 = \$0.12 = 0.01% of $(TVC)'^*$

$$\text{Model Effectiveness} = \left[(\$9.10 - \$0.12)/(\$9.10)\right] \cdot (100\%) = 99\%$$

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